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On classification of soliton solutions of multicomponent nonlinear evolution equations

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Abstract
We consider several ways of how one could classify the various types of soliton solutions related to multicomponent nonlinear evolution equations which are solvable by the inverse scattering method for the generalized Zakharov–Shabat system related to a simple Lie algebra $\mathfrak{g}$. In doing so we make use of the fundamental analytic solutions, the Zakharov–Shabat dressing procedure, the reduction technique and other tools characteristic for that method. The multicomponent solitons are characterized by several important factors: the subalgebras of $\mathfrak{g}$ and the way these subalgebras are embedded in $\mathfrak{g}$, the dimension of the corresponding eigensubspaces of the Lax operator L, as well as by additional constraints imposed by reductions.

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1. Introduction

Since the time when the inverse scattering transform was invented the number of integrable nonlinear evolution equations (NLEE) has been increasing significantly [1–3]. There exist different approaches in analyzing these equations: constructing the spectral theory of the so-called recursion operators from Lax scattering operators, studying whether there exist higher order integrals of motion and so on.

Along with the number of integrable equations the diversity of 'species' of soliton solutions has bloomed up: breathers, topological solitons, trapons, boomerons, etc. Thus the analogous problem of classifying solutions to nonlinear evolution equations and soliton solutions, in particular, becomes more and more important. It is our opinion that the mentioned problem is still waiting for its solution. Even for some of the best known NLEE such as the N-wave equations, the multicomponent nonlinear Schrödinger equations, multicomponent modified

KdV equations, etc only the simplest types of soliton solutions are known. Our aim is to outline several criteria for the classification of the soliton solutions.

We use the term 'soliton solution' as a special solution to a given NLEE which is solvable by the inverse scattering method [1–3], i.e. it allows a zero curvature representation

$$[L(\lambda), M(\lambda)] = 0, \tag{1}$$

where $L(\lambda)$ and $M(\lambda)$ are two linear differential operators. In what follows we take them to be first-order matrix differential operators,

$$L\psi(x,t,\lambda) \equiv \mathrm{i}\partial_x\psi + U(x,t,\lambda)\psi(x,t,\lambda),$$
$$M\psi(x,t,\lambda) \equiv \mathrm{i}\partial_t\psi + V(x,t,\lambda)\psi(x,t,\lambda).$$

The soliton solutions are related to a set of several discrete eigenvalues of the Lax operator L. Therefore one first has to describe all configurations of discrete eigenvalues of L, see [4]. The next step in classifying the types of one-soliton solutions is related to the study of their internal degrees of freedom.

In order to make the problem not too difficult we will specify L to be the operator for the generalized Zakharov–Shabat system

$$L(\lambda)\psi(x,\lambda) \equiv \mathrm{i}\partial_x\psi + (q(x) - \lambda J)\psi(x,\lambda) = 0,$$

where we take the potential an $q(x,t)$ to be an $n \times n$ matrix-valued smooth function of x tending to zero sufficiently rapid as $x \to \pm\infty$. We also restrict J to be a real constant diagonal matrix with different eigenvalues.

For simplicity we consider the class of Lax operators of Zakharov–Shabat type in most cases with real-valued J. In doing this we will be using the dressing method [5–7], one of the best known methods for constructing reflectionless potentials and soliton solutions. This paper is intended as a natural continuation of the work [8] published several years ago by two of the authors.

In section 2, we outline preliminary facts about the generalized Zakharov–Shabat systems related to the $sl(n)$ algebras. We also review the basic facts for the $sl(2)$ soliton solutions, and especially the topological and the breather solutions.

In section 3, we treat the different types of one-soliton solutions for the $sl(n)$ Zakharov–Shabat systems starting with $n = 3$. Along with the well-known soliton solutions obtained by Zakharov and Manakov [9] and Kaup [10] we show that the 3-wave equations with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry allow doublet and quadruplet soliton solutions (analogies to the topological and breather solutions mentioned above). Next we analyze N-wave equations related to $sl(n)$ algebra with $n = 5$; of course nearly all formulae can easily be generalized for any n. Here we use generic projectors of rank $s \geqslant 1$ and explain why it is enough to consider only $s \leqslant [n/2]$ where $[n/2]$ is the entire part of $n/2$. We also demonstrate that using special choices for the polarization vectors defining the projector we can get one-soliton solutions $q_{1\mathrm{s}}(x,t)$ taking values in a subalgebra of $sl(n)$. The simplest nontrivial type of solitons can be related to the subalgebra $sl(2)$; we call them typical $sl(2)$ solitons. Our next observation is that the $sl(2)$ subalgebra can be embedded in several different inequivalent ways into $sl(5)$. For example we can embed a spin J representation of $sl(2)$, $J \leqslant 2$ into $sl(5)$; we call them spin J $sl(2)$ solitons. The corresponding construction is done by using the corresponding symmetrized tensor products of the $sl(2)$ dressing factor. The next step is to consider typical $sl(3)$ and $sl(4)$ solitons. Of course, like for the $sl(3)$ algebra, using additional $\mathbb{Z}_2$-symmetries one can obtain N-wave-type equations allowing doublet and quadruplet solitons.

In section 4, we analyze the structure of the eigenfunctions of $L(\lambda)$ corresponding to the different types of solitons.

In section 5, we discuss the effects of several types of reductions [11] on the different types of one-soliton solutions. The first $\mathbb{Z}_2$-reduction does not affect the spectral parameter λ and is produced using the exterior automorphism of $sl(5)$. Its effect is that now $q(x,t) - \lambda J$ takes values in the subalgebra $so(5) \subset sl(5)$. Here we briefly recall the soliton solutions of the $so(5)$ 4-wave equations we derived in [12]. Next we apply an additional pair of $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions we obtain a 4-wave system allowing doublet and quadruplet solutions. At the end of this section we analyze the NLEE related to the symmetric spaces [13–15] of **BD.I** type; the corresponding J in the Lax pair is dual to the vector e_1 in the root space. One easily checks that for such J the N-wave equations become linear. However there are multicomponent mKdV-type equations which may be called vector mKdV equations. Such equations have been shown to possess higher symmetries and have been related to Jordan algebras [16]. The simplest nontrivial example is a three-component mKdV related to the algebra $so(5)$. We apply an additional $\mathbb{Z}_2$-reduction on it using an automorphism related to a specific Weyl reflection and obtain a two-component vector mKdV which to the best of our knowledge is new. We end the section by deriving for it the doublet and quadruplet solutions.

In section 6 we summarize our results and discuss their possible generalizations. In the appendices, we collect some technical details about the symmetric spaces of **BD.I** type and about the spin J representations of $sl(2)$.

2. Preliminaries

2.1. The generalized Zakharov–Shabat system related to $sl(n)$

In this section, we shall outline some basic features of the mathematical machinery we are about to use for the classification of soliton solutions.

Integrability or more precisely S-integrability of a NLEE means that the NLEE can be presented as a zero curvature condition

$$[L(\lambda), M(\lambda)] = 0 \tag{2}$$

of two first-order linear matrix differential operators $L(\lambda)$ and $M(\lambda)$ of the form

$$L\psi(x,t,\lambda) \equiv (\mathrm{i}\partial_x + U(x,t,\lambda))\psi(x,t,\lambda) = 0, \tag{3}$$

$$M\psi(x,t,\lambda) \equiv (\mathrm{i}\partial_t + V(x,t,\lambda))\psi(x,t,\lambda) = \psi(x,t,\lambda)C(\lambda). \tag{4}$$

The potentials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are typically chosen as elements of some semisimple Lie algebra g (the fundamental solutions $\psi(x,t,\lambda)$ belong to the corresponding Lie group G). We shall mainly deal with the algebra $sl(n)$.

Remark 1. The compatibility condition (2) means that the Lax operators L and M possess the same eigenfunctions. The matrix $C(\lambda)$ depends on the definition of Jost solutions.

Since the compatibility condition (2) must hold true identically with respect to λ one can verify that

$$\mathrm{i}\partial_x V - \mathrm{i}\partial_t U + [U(x,t,\lambda), V(x,t,\lambda)] = 0 \tag{5}$$

and it is valid for any choice of $C(\lambda)$. For simplicity we shall restrict our considerations on scattering operators of the Zakharov–Shabat type (GZS)

$$L(\lambda)\psi(x,t,\lambda) \equiv (\mathrm{i}\partial_x + q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \tag{6}$$

The matrix J is a real traceless diagonal matrix, i.e. a real Cartan element of $sl(n)$, while $q(x,t)$ is a matrix with zero diagonal elements. Since J is a real matrix one can introduce an ordering of its elements $J_1 > J_2 > \cdots > J_n$. By carrying out a gauge transformation which commutes with J, we can always take $q(x,t)$ to be of the form $q(x,t) = [J, Q(x,t)]$, i.e. $q_{jj} \equiv 0$. The linear subspace in $sl(n)$ of matrix-valued functions $q(x,t) = [J, Q(x,t)]$ is known in the literature to be the co-adjoint orbit of g passing through J. The co-adjoint orbits can be supplied in a natural way with a non-degenerate symplectic structure which makes them natural choices for the phase spaces $\mathcal{M}_J$ of the corresponding NLEE.

The class of NLEE related to $L(\lambda)$ are systems of equations for the multicomponent functions $Q(x,t)$, which may be written in the compact form [9, 10, 17, 18]

$$\mathrm{i}\partial_t Q + 2\sum_{k=1}^{4} \Lambda^k [H_k, Q(x,t)] = 0, \tag{7}$$

where H_k, tr $H_k = 0$ are constant diagonal matrices and $f(\lambda) = \sum_{k=1} \lambda^k H_k$ is the dispersion law of the NLEE. Here and below we define

$$(\mathrm{ad}_J X)_{ks} \equiv ([J, X])_{ks} = (J_k - J_s) X_{ks}, \qquad \left(\mathrm{ad}_J^{-1} X\right)_{ks} = \frac{X_{ks}}{J_k - J_s}, \tag{8}$$

for all $X \in \mathcal{M}_J$, i.e. $X_{kk} = 0$. The operator Λ is either one of the recursion operators $\Lambda_\pm$, acting on the space $\mathcal{M}_J$ of $n \times n$ off-diagonal matrix-valued functions as follows:

$$\Lambda_\pm X \equiv \mathrm{ad}_J^{-1}\left(\mathrm{i}\partial_x X + P_0[q, X] + \mathrm{i}\sum_{k=1}^{5} [Q, E_{k,k}] \int_{\pm\infty}^{x} \mathrm{d}y\, \mathrm{tr}([Q, X], E_{k,k})\right). \tag{9}$$

where P_0 is the projector $\mathrm{ad}_J^{-1}\mathrm{ad}_J(E_{k,k})_{jl} = \partial_{k,J}\partial_{J,l}$. Choosing $H_1 = I = \mathrm{diag}(I_1, \ldots, I_n) \neq \mathbb{1}$, so that the dispersion law $f(\lambda) = -\lambda I$ is a linear function of λ we get a system, generalizing the well-known N-wave equation

$$\mathrm{i}[J, Q_t] - \mathrm{i}[I, Q_x] - [[J, Q], [I, Q]] = 0, \tag{10}$$

which contains $N = n(n-1)$ complex-valued functions $Q_{ij}(x,t)$.

In order to describe the soliton solutions we shall use the so-called dressing procedure [7]. For that purpose we need some basic facts on the direct scattering problem for L.

Let $\psi_\pm(x,t,\lambda)$ are two fundamental solutions of the GZS system (6). If they satisfy the requirement

$$\lim_{x\to\pm\infty} \psi_\pm(x,t,\lambda)\, \mathrm{e}^{\mathrm{i}\lambda J x} = \mathbb{1} \tag{11}$$

they shall be called Jost solutions. The Jost solutions are interrelated via

$$\psi_-(x,t,\lambda) = \psi_+(x,t,\lambda) T(t,\lambda), \tag{12}$$

where $T(t,\lambda)$ is called a scattering matrix. The scattering matrix is x-independent and its time evolution is driven by the linear equation

$$\mathrm{i}\partial_t T + [f(\lambda), T] = 0 \qquad \Rightarrow \qquad T(t,\lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t} T(0,\lambda)\, \mathrm{e}^{-\mathrm{i}f(\lambda)t}. \tag{13}$$

Equation (13) shows how one can recover the time evolution of the scattering data. It is used to solve Cauchy's problem for NLEE as it is displayed below,

$$q(t=0) \to L(t=0) \to T(t=0) \to T(t) \to L(t) \to q(t). \tag{14}$$

Due to this we shall skip the t-dependence of all functions (potentials, fundamental solutions, etc) regarding them at a fixed moment $t = t_0$.

The set of matrix elements of $T(\lambda)$ must satisfy a number of relations. Indeed, they are uniquely determined by $Q(x)$, i.e. by $n(n-1)$ complex functions of x, so it seems natural that there should not be more than $n(n-1)$ independent functions among $T_{jk}(\lambda)$ for λ on the real axis. Of course, $T(\lambda)$ must satisfy the 'unitarity' condition $\det T(\lambda) = 1$. The rest of these relations follow from the analyticity properties of certain combinations of the matrix elements of $T(\lambda)$. These analyticity properties must follow naturally from the corresponding fundamental analytic solutions (FAS) $\chi^{\pm}(x,\lambda)$.

The Jost solutions are well defined only for $\lambda \in \mathbb{R}$, i.e. they do not have necessarily analytic properties beyond the real axis. This can be easily seen if one reformulates the problem (6) in terms of a Volterra-type integral equation

$$\xi_{\pm}(x,\lambda) = \mathbb{1} + \mathrm{i}\int_{\pm\infty}^{x} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}\lambda J(y-x)} q(y)\xi_{\pm}(y,\lambda)\, \mathrm{e}^{\mathrm{i}\lambda J(x-y)}, \tag{15}$$

where $\xi_{\pm}(x,\lambda) = \psi_{\pm}(x,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda Jx}$ represents another set of fundamental solutions but this time to the linear problem

$$\mathrm{i}\partial_x\xi(x,\lambda) + q(x)\xi(x,\lambda) - \lambda[J,\xi(x,\lambda)] = 0.$$

It is easy to see that only the first and the last columns of $\psi_{+}(x,\lambda)$ and $\psi_{-}(x,\lambda)$ allow analytic extensions in λ off the real axis; generally the other columns do not have analyticity properties. Nevertheless it is possible to introduce FAS [3, 19]. Taking into account the ordering introduced above one is able to construct new fundamental solutions

$$\xi^{\pm}_{kl}(x,\lambda) = \begin{cases} \delta_{kl} + \mathrm{i}\displaystyle\int_{\pm\infty}^{x} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}\lambda(J_k-J_l)(y-x)}(q\xi^{\pm})_{kl}(y,\lambda), & k \leqslant l, \\ \mathrm{i}\displaystyle\int_{\mp\infty}^{x} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}\lambda(J_k-J_l)(y-x)}(q\xi^{\pm})_{kl}(y,\lambda), & k > l \end{cases} \tag{16}$$

to possess analytic properties in the half planes $\mathbb{C}_{\pm}$ of the spectral parameter. This definition can be rewritten using the Gauss factors of the scattering matrix T

$$\chi^{\pm}(x,\lambda) = \psi_{-}(x,\lambda)\mathbb{S}^{\pm}(\lambda) = \psi_{+}(x,\lambda)\mathbb{T}^{\mp}(\lambda), \tag{17}$$

where $T(\lambda) = \mathbb{T}^{\mp}(\lambda)(\mathbb{S}^{\pm}(\lambda))^{-1}$ and $\chi^{\pm}(x,\lambda) = \xi^{\pm}(x,\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda Jx}$. The matrix elements of $\mathbb{T}^{\pm}(\lambda)$ and $\mathbb{S}^{\pm}(\lambda)$ can be expressed in terms of the minors of $T(\lambda)$. Here we note that their diagonal elements can be given by

$$\mathbb{S}^{+}_{jj}(\lambda) = m^{+}_{j-1}(\lambda), \qquad \mathbb{T}^{-}_{jj}(\lambda) = m^{+}_{j}(\lambda), \tag{18}$$

$$\mathbb{T}^{+}_{jj}(\lambda) = m^{-}_{n-j}(\lambda), \qquad \mathbb{S}^{-}_{jj}(\lambda) = m^{-}_{n+1-j}(\lambda), \tag{19}$$

where $m^{\pm}_0 = m^{\pm}_n = 1$ and by $m^{+}_k(\lambda)$ (resp. $m^{-}_k(\lambda)$) we have denoted the upper (resp. lower) principal minors of $T(\lambda)$ of order k, e.g.,

$$m^{+}_k(\lambda) = \begin{Bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{Bmatrix}, \qquad k = 1,\dots,n \tag{20}$$

$$m^{-}_k(\lambda) = \begin{Bmatrix} n-k+1 & n-k+2 & \dots & n \\ n-k+1 & n-k+2 & \dots & n \end{Bmatrix}_{T(\lambda)}, \tag{21}$$

$$\begin{Bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{Bmatrix}_{T(\lambda)} \equiv \det \begin{pmatrix} T_{i_1j_1} & T_{i_1j_2} & \dots & T_{i_1j_k} \\ T_{i_2j_1} & T_{i_2j_2} & \dots & T_{i_2j_k} \\ \vdots & \vdots & \ddots & \vdots \\ T_{i_kj_1} & T_{i_kj_2} & \dots & T_{i_kj_k} \end{pmatrix}. \tag{22}$$

As a consequence of the analyticity of the FAS, it follows that the minors $m^{+}_k(\lambda)$ (resp. $m^{-}_k(\lambda)$) are analytic functions for $\lambda \in \mathbb{C}_{+}$ (resp. for $\lambda \in \mathbb{C}_{-}$).

One can construct the kernel of the resolvent of $L(\lambda)$ in terms of the FAS [17, 18] from which it follows that the resolvent has poles for all $\lambda_k^\pm$ which happen to be zeroes of any of the minors $m_k^\pm(\lambda)$. Therefore what we have now is that each of the minors $m_k^\pm(\lambda)$ may be considered to be an analog of the Evans function, and thus now, there is more than one Evans function, with each Evans function in a one-to-one correspondence with one of the minors, $m_k^\pm(\lambda)$.

There exist different methods to solve a NLEE possessing a Lax representation: Gel'fand–Levitan–Marchenko integral equation, Hirota method, dressing method, etc. We shall use the dressing Zakharov–Shabat method [7]. Let $\psi_0(x,\lambda)$ be a fundamental solution of Zakharov–Shabat's system with a known potential $U_0(x,\lambda) = q_0(x) - \lambda J$. Consider a new function $\psi(x,\lambda) = u(x,\lambda)\psi_0(x,\lambda)$ which is a solution to Zakharov–Shabat's problem with some potential $q(x) - \lambda J$ to be found. This requires that $u(x,\lambda)$ satisfies

$$\mathrm{i}\partial_x u + qu - uq_0 - \lambda[J,u] = 0. \tag{23}$$

The dressing procedure transforms the Jost solutions $\psi_{\pm,0}(x,\lambda)$, the scattering matrix $T_0(\lambda)$ and the fundamental solution $\chi_0^\pm(x,\lambda)$ of the generalized Zakharov–Shabat system with a potential $U_0(x,\lambda)$ in the following fashion:

$$\psi_\pm(x,\lambda) = u(x,\lambda)\psi_{\pm,0}(x,\lambda)u_\pm^{-1}(\lambda), \tag{24}$$

$$T(\lambda) = u_+(\lambda)T_0(\lambda)u_-^{-1}(\lambda), \tag{25}$$

$$\chi^\pm(x,\lambda) = u(x,\lambda)\chi_0^\pm(x,\lambda)u_-^{-1}(\lambda). \tag{26}$$

The normalizing factors $u_\pm(\lambda) = \lim_{x\to\pm\infty} u(x,\lambda)$ ensures the proper asymptotics of the dressed solutions $\psi_\pm(x,\lambda)$.

Our aim is by using the structure of the dressing factor to try to classify the soliton solutions of NLEE. A classical ansatz [3] for $u(x,\lambda)$ is the following one:

$$u(x,\lambda) = \mathbb{1} + (c(\lambda) - 1)P(x), \qquad c(\lambda) = \frac{\lambda - \lambda^+}{\lambda - \lambda^-}, \tag{27}$$

where P is a projector ($P^2 = P$) which can be expressed via the fundamental analytic solutions (FAS) and λ^+ (resp. λ^-) is an arbitrary complex number in the upper (resp. lower) half plane $\mathbb{C}_+$ (resp. $\mathbb{C}_-$). After applying the dressing procedure $\lambda^\pm$ become a pair of discrete eigenvalues for the dressed Lax operator. The projector $P(x)$ determines the corresponding eigensubspaces—the rank s of P equals the dimension of the eigensubspaces. If we introduce two sets, each containing s vectors

$$|n_1\rangle, \ldots, |n_s\rangle; \qquad \langle m_1|, \ldots, \langle m_s|,$$

we can write P as follows:

$$P = \sum_{a,b=1}^{s} |n_a\rangle \widehat{M}_{ab}\langle m_b|, \qquad M_{ab} = \langle m_a|n_b\rangle, \qquad \widehat{M} \equiv M^{-1}. \tag{28}$$

Obviously the two sets of vectors are the left and right eigenvectors of P, namely

$$P|n_a\rangle = |n_a\rangle, \qquad \langle m_b|P = \langle m_b|, \tag{29}$$

for all $a, b = 1, \ldots, s$. In the simplest case when $\operatorname{rank} P = 1$ it reads

$$P(x) = \frac{|n(x)\rangle\langle m(x)|}{\langle m(x)|n(x)\rangle}, \tag{30}$$

where

$$|n(x)\rangle = \chi_0^+(x,\lambda^+)|n_0\rangle, \qquad \langle m(x)| = \langle m_0|(\chi_0^-(x,\lambda^-))^{-1}. \tag{31}$$

By taking the limit $\lambda \to \infty$ in equation (23) we obtain an interrelation between the seed solution q_0 and the new one

$$q = q_0 + (\lambda^- - \lambda^+)[J, P]. \tag{32}$$

Thus starting from a known solution of the NLEE we can find another solution by simply dressing it with some factor $u(x, \lambda)$. An important particular case is when $q_0 = 0$. The dressed solution is called a one-soliton solution. The fundamental analytic solution in the soliton case is given by a plane wave $\chi_0^{\pm}(x, \lambda) = \exp(-\mathrm{i}\lambda J x)$. Repeating the same procedure one derives step by step the multisoliton solution of the corresponding NLEE, i.e.,

$$0 \to q^{1\mathrm{s}} \to q^{2\mathrm{s}} \to \cdots \to q^{n\mathrm{s}}. \tag{33}$$

Many integrable equations correspond to Lax operators that obey some additional symmetry conditions of algebraic nature. That is why it is worthwhile to outline some aspects of the theory of such Lax operators.

Let an action of a discrete group G_R to be referred to as a reduction group be given on the set of fundamental solutions to the generalized Zakharov–Shabat system (6) as follows:

$$\tilde{\psi}(x, \lambda) = K\psi(x, k(\lambda))K^{-1}, \tag{34}$$

where $k : \mathbb{C} \to \mathbb{C}$ is a conformal map. This action yields another action on the potential in the scattering operator L,

$$KU(x, k(\lambda))K^{-1} = U(x, \lambda). \tag{35}$$

A common case is when $G_R = \mathbb{Z}_2$. Then the action of $\mathbb{Z}_2$ might involve external automorphisms of $SL(n)$ as well,

$$\tilde{\psi}(x, \lambda) = K(\psi^T(x, k_1(\lambda)))^{-1}K^{-1} \qquad \Rightarrow \qquad KU^T(x, k_1(\lambda))K^{-1} = -U(x, \lambda), \tag{36}$$

$$\tilde{\psi}(x, \lambda) = K\psi^*(x, k_2(\lambda))K^{-1} \qquad \Rightarrow \qquad KU^*(x, k_2(\lambda))K^{-1} = -U(x, \lambda). \tag{37}$$

In particular, if $\mathbb{Z}_2$ acts trivially on the complex plane of the spectral parameter, i.e. $k = \mathrm{i}d$, then the symmetry condition (36) may restricts the potential $U(x, \lambda)$ to a certain subalgebra of $sl(n)$. For example, suppose $K^T = S_0\widehat{K}\widehat{S}_0$ then $U(x, \lambda)$ belongs to the orthogonal algebra[3] $so(n)$. The existence of a $\mathbb{Z}_2$ reduction requires a modification of the dressing factor $u(x, \lambda)$ as follows:

$$u(x, \lambda) = \mathbb{1} + (c(\lambda) - 1)P(x) + \left(\frac{1}{c(\lambda)} - 1\right)\overline{P}(x), \tag{38}$$

where $P(x)$ is a projector of rank 1,

$$P(x) = \frac{|n(x)\rangle\langle m(x)|}{\langle m(x)|n(x)\rangle}, \qquad \overline{P}(x) = KS_0P^T(x)S_0^{-1}K^{-1}. \tag{39}$$

The projector itself can be expressed through the FAS $\chi^{\pm}(x, \lambda)$,

$$|n(x)\rangle = \chi_0^+(x, \lambda^+)|n_0\rangle, \qquad \langle m(x)| = \langle m_0|(\chi_0^-(x, \lambda^-))^{-1}. \tag{40}$$

[3] Here we use a slightly modified definition of the orthogonal algebras, see appendix A where the definition and the explicit form of the matrix S_0 are given.

2.2. The Zakharov–Shabat system and $sl(2)$ solitons

The best known examples of NLEE are related to the Zakharov–Shabat system which is associated with the $sl(2)$ algebra as follows:

$$\begin{aligned} L\psi(x,t,\lambda) &\equiv (\mathrm{i}\partial_x + q(x,t) - \lambda\sigma_3)\psi(x,t,\lambda), \\ q(x,t) &= q^+\sigma_+ + q^-\sigma_- = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix}, \end{aligned} \tag{41}$$

where $\sigma_\pm = (\sigma_1 \pm \mathrm{i}\sigma_2)/2$ and σ_1, σ_2 and σ_3 are the Pauli matrices.

The class of NLEE for the functions $q^\pm(x,t)$ related to (41) can be written in the compact form [20–22],

$$\mathrm{i}\sigma_3\partial_t q + 2f(\Lambda)q(x,t) = 0, \tag{42}$$

where $f(\lambda)$ is the dispersion law of the NLEE and Λ is one of the recursion operators, acting on the space $\mathcal{M}_0$ of off-diagonal matrix-valued functions as follows:

$$\Lambda_\pm X \equiv \frac{\mathrm{i}}{4}[\sigma_3, \partial_x X] + \frac{\mathrm{i}}{2}q(x)\int_{\pm\infty}^{x} \mathrm{d}y\, \mathrm{tr}(q(y), [\sigma_3, X(y)]). \tag{43}$$

The simplest nontrivial example of NLEE is related to a dispersion law of the type $f(\lambda) = -2\lambda^2$. This is the nonlinear Schrödinger equation

$$\begin{aligned} &\mathrm{i}q_t^+ + q_{xx}^+ + 2(q^+(x,t))^2 q^-(x,t) = 0, \\ &\mathrm{i}q_t^- - q_{xx}^- - 2(q^-(x,t))^2 q^+(x,t) = 0. \end{aligned} \tag{44}$$

Another well-known example is provided by a cubic dispersion law $f(\lambda) = 4\lambda^3$, one gets the system

$$\begin{aligned} &q_t^+ + q_{xxx}^+ + 6q^+(t)q^-(x,t)q_x^+ = 0, \\ &q_t^- + q_{xx}^- + 6q^-(x,t)q^+(x,t)q_x^- = 0, \end{aligned} \tag{45}$$

directly linking to the Korteweg de Vries equation.

As we discussed in the previous section the scattering theory is based on introducing Jost solutions of $L(\lambda)$, scattering matrix, fundamental solutions, etc. In the $sl(2)$ case the Jost solutions are 2×2 matrix-valued solutions defined by an analog of (11) where the matrix J is simply substituted by σ_3. Then one introduces the scattering matrix $T(\lambda, t)$ by

$$T(\lambda,t) \equiv (\psi_+(x,t,\lambda))^{-1}\psi_-(x,t,\lambda) = \begin{pmatrix} a^+(\lambda) & -b^-(\lambda,t) \\ b^+(\lambda,t) & a^-(\lambda) \end{pmatrix}, \tag{46}$$

which is x-independent. The t-dependence of the scattering matrix is driven by

$$\mathrm{i}\partial_t T + [f(\lambda)\sigma_3, T(\lambda,t)] = 0. \tag{47}$$

Thus, if $q^\pm(x,t)$ satisfy the system of equations (42) we get

$$\partial_t a^\pm(\lambda) = 0, \qquad \mathrm{i}\partial_t b^\pm(\lambda) \mp 2f(\lambda)b^\pm(\lambda) = 0. \tag{48}$$

The matrix elements of $T(\lambda, t)$ are not independent. They satisfy the 'unitarity' condition $\det T(\lambda) \equiv a^+a^- + b^+b^- = 1$. Besides the diagonal elements a^+ and a^- allow analytic extension with respect to λ in the upper and lower complex λ-plane respectively. In fact the minimal set of scattering data which uniquely determines both the scattering matrix and the corresponding potential $q(x)$ consists of two types of variables: (i) the reflection coefficients $\rho^\pm(\lambda) = b^\pm/a^\pm$ defined for real $\lambda \in \mathbb{R}$ and (ii) a discrete set of scattering data including the discrete eigenvalues $\lambda_k^\pm \in \mathbb{C}_\pm$ and the constants $C_k^\pm$ which determine the norm of the corresponding Jost solutions [23].

A simple analysis shows that the first column of ψ_+ allows analytic continuation in the lower half plane of the spectral parameter while the last one in the upper half plane (for ψ_- the opposite holds true)

$$\psi_+(x,t,\lambda) = \left|\psi_+^-, \psi_+^+\right|, \qquad \psi_-(x,t,\lambda) = \left|\psi_-^+, \psi_-^-\right|. \tag{49}$$

The superscripts $\pm$ in the columns of the Jost solutions refer to their analyticity properties while the subscripts $\pm$ refer to different Jost solutions (with different limits of x). The fundamental analytic solutions are constructed in the following manner:

$$\chi^+(x,t,\lambda) = \left|\psi_-^+, \psi_+^+\right|, \qquad \chi^-(x,t,\lambda) = \left|\psi_+^-, \psi_-^-\right|. \tag{50}$$

The functions $a^\pm(\lambda) = \det \chi^\pm(x,\lambda)$ are known as the Evans functions [6, 24] of the system $L(\lambda)$. Their importance comes from the fact that they are t-independent (see (48)), and therefore they (or rather $\ln a^\pm$) can be viewed as generating functionals of the (local) integrals of motion. In addition it is known that their zeroes determine the discrete eigenvalues of $L(\lambda)$,

$$a^+\left(\lambda_k^+\right) = 0, \qquad \lambda_k^+ \in \mathbb{C}_+; \qquad a^-\left(\lambda_k^-\right) = 0, \qquad \lambda_k^- \in \mathbb{C}_-. \tag{51}$$

One can define the soliton solutions of the NLEE as those for which $\rho^\pm(\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Thus the soliton solutions of (42) are parametrized by the discrete eigenvalues and the constants $C_k^\pm$ whose t-dependence is determined from

$$\frac{d\lambda_k^\pm}{dt} = 0, \qquad i\frac{dC_k^\pm}{dt} \mp 2f_k^\pm C_k^\pm = 0, \qquad f_k^\pm = f\left(\lambda_k^\pm\right). \tag{52}$$

In fact we will analyze the various possible types of one-soliton solutions; in our case they are determined by one *pair* of discrete eigenvalues $\lambda^\pm \in \mathbb{C}_\pm$ and one *pair* of norming constants $C^\pm$. Thus for (42) we get just one type of one-soliton solutions. In order to derive its explicit form we shall use the dressing Zakharov–Shabat method [7]. In our case the dressing factor $u(x,t,\lambda)$ is given by a 2×2 matrix of the form (27) where P is a projector of rank 1 (see (30)). Then the following relations hold

$$P|n(x,t)\rangle = |n(x,t)\rangle, \qquad |n(x,t)\rangle = \begin{pmatrix} n_1(x,t) \\ n_2(x,t) \end{pmatrix}, \tag{53}$$

$$\langle m(x,t)|P(x,t) = \langle m(x,t)|, \qquad \langle m(x,t)| = (m_1(x,t), m_2(x,t)). \tag{54}$$

The transmission coefficients are transformed by the dressing procedure as follows:

$$a^+(\lambda) = c(\lambda)a_0^+(\lambda), \qquad a^-(\lambda) = \frac{a_0^-(\lambda)}{c(\lambda)}. \tag{55}$$

The $sl(2)$ analog of (32) reads

$$q(x,t) - q_0(x,t) = -(\lambda^+ - \lambda^-)[\sigma_3, P(x,t)]. \tag{56}$$

By applying the above formulae to properly chosen constant vectors $|n_0\rangle$ and $\langle m_0|$ we can construct the eigenvectors of $P(x,t)$ and as a result, obtain $P(x,t)$ explicitly. It then remains only to insert it into (56) in order to obtain the corresponding potential $q(x,t)$ explicitly. It can be proved that the spectrum of $L(\lambda)$ will differ from the spectrum of $L_0(\lambda)$ only by an additional pair of discrete eigenvalues located at $\lambda^\pm \in \mathbb{C}_\pm$.

A pure soliton solution is obtained by assuming $q_0(x,t) = 0$; as a result we have

$$\begin{gathered}
|n(x,t)\rangle = e^{-i(x\lambda^+ + f^+t)\sigma_3}|n_0\rangle, \qquad \langle m(x,t)| = \langle m_0|\, e^{i(x\lambda^- - f^-t)\sigma_3}, \\
P(x,t) = \frac{1}{2\cosh\Phi_0(x,t)} \begin{pmatrix} e^{\Phi_0(x,t)} & \kappa_2\, e^{-i\Phi(x,t)} \\ \frac{1}{\kappa_2}\, e^{i\Phi(x,t)} & e^{-\Phi_0(x,t)} \end{pmatrix} \\
\Phi_0(x,t) = -i(\lambda^+ - \lambda^-)x + i(f^+ - f^-)t - \ln\kappa_1, \\
\Phi(x,t) = (\lambda^+ + \lambda^-)x - (f^+ + f^-)t,
\end{gathered} \tag{57}$$

where $f^{\pm}$ and the constants κ_1 and κ_2 are given by

$$f^{\pm} = f(\lambda^{\pm}), \qquad \kappa_1 = \sqrt{\frac{n_{0,1}m_{0,1}}{n_{0,2}m_{0,2}}}, \qquad \kappa_2 = \sqrt{\frac{n_{0,1}m_{0,2}}{n_{0,2}m_{0,1}}}. \tag{58}$$

Then the corresponding one-soliton solution takes the form

$$q^{+}(x,t) = -\frac{\kappa_2(\lambda^{+}-\lambda^{-})\,\mathrm{e}^{-\mathrm{i}\Phi(x,t)}}{\cosh\Phi_0(x,t)}, \qquad q^{-}(x,t) = \frac{\kappa_2(\lambda^{+}-\lambda^{-})\,\mathrm{e}^{\mathrm{i}\Phi(x,t)}}{\kappa_2\cosh\Phi_0(x,t)}. \tag{59}$$

Remark 2. The eigenvalues $\lambda^{\pm}$ are two independent complex numbers; therefore, in the denominator in (57) we get cosh of complex argument. This function vanishes whenever its argument becomes equal to $\mathrm{i}(\pi/2 + p\pi)$ for some integer p and the generic solitons of (42) may have singularities for finite x and t.

One way to avoid these singularities is to impose on the Zakharov–Shabat system an involution, i.e. if we constrain the potential $q_0(x,t)$ by

$$q(x,t) = q^{\dagger}(x,t), \qquad \text{i.e.} \qquad q^{+} = (q^{-})^{*} = u(x,t). \tag{60}$$

Such constraint reduces (42) to NLEE for the single function $u(x,t)$; the second equation of the system becomes consequence of the first one. As a result (44) becomes the NLS equation

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u(x,t) = 0, \tag{61}$$

while (45) goes into the MKdV-type equation

$$u_t + u_{xxx} + 6|u(x,t)|^2 u_x = 0. \tag{62}$$

This involution imposes constraints on all the scattering data; in particular we have

$$a^{+}(\lambda) = (a^{-}(\lambda^{*}))^{*}, \qquad b^{+}(\lambda) = (b^{-}(\lambda^{*}))^{*}. \tag{63}$$

From the first relation above we find that the zeroes of the functions $a^{\pm}(\lambda)$ which are the eigenvalues of $L_0(\lambda)$ must satisfy

$$\lambda^{+} = (\lambda^{-})^{*} = \mu + \mathrm{i}\nu, \qquad C^{+} = (C^{-})^{*}, \qquad P(x,t) = P^{\dagger}(x,t). \tag{64}$$

So now the one-soliton solution corresponds to a pair of eigenvalues which must be mutually conjugated pairs.

As a result we find that the expression for $P(x,t)$ and the one for the one-soliton solution simplifies to

$$\begin{aligned} P(x,t) &= \frac{1}{2\cosh\Phi_{00}(x,t)}\begin{pmatrix} \mathrm{e}^{\Phi_{00}(x,t)} & \mathrm{e}^{-\mathrm{i}\Phi_{01}(x,t)} \\ \mathrm{e}^{\mathrm{i}\Phi_{01}(x,t)} & \mathrm{e}^{-\Phi_{00}(x,t)} \end{pmatrix} \\ \Phi_{00}(x,t) &= 2\nu x - 2ht - \ln\left|\frac{n_0^1}{n_0^2}\right|, \\ \Phi_{01}(x,t) &= 2\mu x - 2gt - \arg n_0^1 + \arg n_0^2, \end{aligned} \tag{65}$$

where

$$\lambda^{\pm} = \mu \pm \mathrm{i}\nu, \qquad f^{\pm} = g \pm \mathrm{i}h. \tag{66}$$

Now both functions $\Phi_{00}(x,t)$ and $\Phi_{01}(x,t)$ become real valued. The denominator now becomes cosh of real argument, so this soliton solution is regular function for all x and t.

One can impose on $q_0(x,t)$ a different involution

$$q(x,t) = -q^{\dagger}(x,t), \qquad \text{i.e.} \qquad q^{+} = -(q^{-})^{*} = u(x,t). \tag{67}$$

However it is well known that under this involution the Zakharov–Shabat system $L(\lambda)$ becomes equivalent to an eigenvalue problem

$$\mathcal{L}\psi(x,t,\lambda) \equiv \mathrm{i}\sigma_3\partial_x\psi + \sigma_3 q(x,t)\psi(x,t,\lambda) = \lambda\psi(x,t,\lambda), \tag{68}$$

where the operator $\mathcal{L}$ is a self-adjoint one, so its spectrum must be on the real λ-axis. But the continuous spectrum of $\mathcal{L}$ fills up the whole real λ-axis, which leaves no room for solitons.

Finally, the Zakharov–Shabat system can be restricted by a third involution, e.g.,

$$q(x,t) = -q^T(x,t), \qquad \text{i.e.} \qquad q^+ = -q^- = -\mathrm{i}w_x. \tag{69}$$

Such involution is compatible only with those NLEE whose dispersion law is odd function $f(\lambda) = -f(-\lambda)$. Therefore it cannot be applied to the NLS equation; applied to the MKdV equation it gives

$$w_{xt} + w_{xxxx} + 6(w_x(x,t))^2 w_{xx} = 0, \tag{70}$$

which can be integrated ones with the result $v = w_x$,

$$v_t + v_{xxx} + 6(v(x,t))^2 v_x = 0, \tag{71}$$

i.e. we get the MKdV equation for the real-valued function $v(x,t)$. It is well known also that the NLEE with dispersion law $f(\lambda) = (2\lambda)^{-1}$ can be explicitly derived under this reduction and comes out to be the famous sine-Gordon equation [25]

$$w_{xt} + \sin(2w(x,t)) = 0. \tag{72}$$

This second involution can be imposed together with the one in (60). The restrictions that it imposes on the scattering data are as follows:

$$a^+(\lambda) = (a^-(\lambda^*))^*, \qquad a^+(\lambda) = (a^-(-\lambda)). \tag{73}$$

Now if λ^+ is an eigenvalue of $L(\lambda)$ then $(\lambda^+)^*$, $-\lambda^+$ and $-(\lambda^+)^*$ must also be eigenvalues. This means that we can have two configurations of eigenvalues:

(i) pairs of purely imaginary eigenvalues

$$\lambda^+ = \mathrm{i}\nu \equiv -(\lambda^+)^*, \qquad \lambda^- = -\mathrm{i}\nu \equiv -(\lambda^-)^*; \tag{74}$$

(ii) quadruplets of complex eigenvalues

$$\begin{aligned} \lambda^+ &= \mu + \mathrm{i}\nu & -(\lambda^+)^* &= -\mu + \mathrm{i}\nu, \\ \lambda^- &= \mu - \mathrm{i}\nu, & -(\lambda^-)^* &= -\mu - \mathrm{i}\nu. \end{aligned} \tag{75}$$

Thus we conclude, that the sine-Gordon and MKdV equations allow *two types* of solitons: type 1 with purely imaginary pairs of eigenvalues and type 2 each corresponding to a quadruplet of eigenvalues. Type 1 solitons are known also as topological solitons, or kinks (for details see [2]). They are parametrized by two real parameters: ν and $|C^+|$ so they have just one degree of freedom corresponding to the uniform motion.

Type 2 solitons are known as the breathers and are parametrized by four real parameters: μ and ν and the real and imaginary parts of C^+. Therefore they have two degrees of freedom: one corresponds to the uniform motion and the second one describes the internal degree of freedom responsible for the 'breathing'.

The purpose of presenting the above well-known facts in the above manner was simply to make it clear that the structure, as well as the number of related parameters which determine what different types of solitons can exist, depend strongly on the type of, and the number of, different involutions that can be imposed on the system.

3. The Generalized Zakharov–Shabat system and $sl(n)$ solitons

3.1. N-wave system related to $sl(3)$

In this subsection, we start with the generic N-wave system related to $sl(3)$ and analyze its reductions. The one-soliton solutions for the generic case and for the typical reductions are well known [9, 10]. Next we impose $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions and derive the corresponding one-soliton solutions. Below we will use a notation which exploits the root structure of $sl(3)$, namely $Q_{kn}k, n = 1, 2$ stands for the component of Q associated with the root $\alpha = k\alpha_1 + n\alpha_2$ expanded over the simple roots $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. Taking into account that convention the generic N-wave system for $sl(3)$ consists of six equations; the first three of them are given by

$$\mathrm{i}(J_1 - J_2)Q_{10,t} - \mathrm{i}(I_1 - I_2)Q_{10,x} + 3kQ_{11}Q_{\overline{01}} = 0, \tag{76}$$

$$\mathrm{i}(J_1 - J_3)Q_{11,t} - \mathrm{i}(I_1 - I_3)Q_{11,x} - 3kQ_{10}Q_{01} = 0, \tag{77}$$

$$\mathrm{i}(J_2 - J_3)Q_{01,t} - \mathrm{i}(I_2 - I_3)Q_{01,x} + 3kQ_{\overline{10}}Q_{11} = 0; \tag{78}$$

the other three equations are obtained from (76) by replacing $J_k \leftrightarrow -J_k$, $I_k \leftrightarrow -I_k$ and $Q_{kn} \leftrightarrow Q_{\overline{kn}}$. We recall that $\sum_{k=1}^{3} J_k = 0$, $\sum_{k=1}^{3} I_k = 0$ and $k = J_1I_2 - I_1J_2$. This system can be solved via a standard dressing procedure [7, 9] with the dressing factor (27). The one-soliton solution obtained that way is given by the following expressions:

$$\begin{aligned} Q_{10}(x,t) &= \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}\,\mathrm{e}^{-\mathrm{i}(\lambda^+ z_1 - \lambda^- z_2)} n_{0,1}m_{0,2}, \\ Q_{11}(x,t) &= \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}\,\mathrm{e}^{-\mathrm{i}(\lambda^+ z_1 - \lambda^- z_3)} n_{0,1}m_{0,3}, \\ Q_{01}(x,t) &= \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}\,\mathrm{e}^{-\mathrm{i}(\lambda^+ z_2 - \lambda^- z_3} n_{0,2}m_{0,3}, \end{aligned} \tag{79}$$

where $z_\sigma = J_\sigma x + I_\sigma t$ and

$$\langle m|n\rangle = \sum_{j=1}^{3} \mathrm{e}^{-\mathrm{i}(\lambda^+ - \lambda^-)z_j} n_{0,j}m_{0,j}. \tag{80}$$

The expressions for the other three fields can be derived from these by executing the following changes $Q_{kn} \leftrightarrow Q_{\overline{kn}}$, $\mathrm{e}^{\mathrm{i}\lambda^+ z_j} \leftrightarrow \mathrm{e}^{-\mathrm{i}\lambda^- z_j}$, $n_{0,j} \leftrightarrow m_{0,j}$.

Next we consider the typical $\mathbb{Z}_2$ reductions of the type

$$K_1U^\dagger(x,\lambda^*)K_1^{-1} = U(x,\lambda), \quad \Rightarrow K_1J^*K_1^{-1} = J, \qquad K_1Q^\dagger K_1^{-1} = -Q, \tag{81}$$

where $K_1 = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3)$, $\epsilon_j^2 = 1$, is an element of the Cartan subgroup $H \subset SL(3)$ which represents an action of $\mathbb{Z}_2$. This results in reducing the number of independent fields since we have

$$Q_{\overline{10}} = -\epsilon_1\epsilon_2 Q_{10}^*, \qquad Q_{\overline{11}} = -\epsilon_1\epsilon_3 Q_{11}^*, \qquad Q_{\overline{01}} = -\epsilon_2\epsilon_3 Q_{01}^*$$

and therefore the number of equations reduces from 6 to 3 as follows:

$$\mathrm{i}(J_1 - J_2)Q_{10,t} - \mathrm{i}(I_1 - I_2)Q_{10,x} - 3k\epsilon_2\epsilon_3 Q_{11}Q_{01}^* = 0, \tag{82}$$

$$\mathrm{i}(J_1 - J_3)Q_{11,t} - \mathrm{i}(I_1 - I_3)Q_{11,x} - 3kQ_{10}Q_{01} = 0, \tag{83}$$

$$\mathrm{i}(J_2 - J_3)Q_{01,t} - \mathrm{i}(I_2 - I_3)Q_{01,x} - 3k\epsilon_1\epsilon_2 Q_{10}^*Q_{11} = 0. \tag{84}$$

The discrete eigenvalues of $\mathbb{Z}_2$-reduced operator L are complex conjugated, i.e. $\lambda^- = (\lambda^+)^* = \mu - i\nu$ and the polarization vectors are interrelated via $|n\rangle = K_1|m\rangle^*$. The one-soliton solution in this case is

$$Q_{10}(x,t) = -\frac{2i\nu\, e^{\nu(z_1+z_2)}}{\langle n^*|K_1|n\rangle}\, e^{-i\mu(z_1-z_2)} n_{0,1}\epsilon_2 n_{0,2}^*, \tag{85}$$

$$Q_{11}(x,t) = -\frac{2i\nu\, e^{-\nu z_2}}{\langle n^*|K_1|n\rangle}\, e^{-i\mu(z_1-z_3)} n_{0,1}\epsilon_3 n_{0,3}^*, \tag{86}$$

$$Q_{01}(x,t) = -\frac{2i\nu\, e^{-\nu z_1}}{\langle n^*|K_1|n\rangle}\, e^{-i\mu(z_2-z_3)} n_{0,2}\epsilon_3 n_{0,3}^*. \tag{87}$$

Remark 3. In general, the denominator (80) of the expressions for the one-soliton solution can possesses zeros for some x and t, i.e. we have singular solitons. The same holds true also for the 3-wave system with the typical involutions for which $K \neq \mathbb{1}$. This is directly related to the well-known 'blow-up' instability [9]. For the typical involution with $K_1 = \mathbb{1}$ the corresponding denominator is a sum real-valued exponentials multiplied by some positive factors which is always positive.

By imposing another $\mathbb{Z}_2$ reduction on the potential $U(x,\lambda)$, namely

$$K_2 U^T(x,-\lambda) K_2^{-1} = -U(x,\lambda), \qquad K_2 J^T K_2^{-1} = J, \qquad K_2 Q^T K_2^{-1} = Q, \tag{88}$$

where $K_2 \in H$ satisfies $[K_1, K_2] = 0$ we obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduced $sl(3)N$-wave system. As a consequence we have a pair of purely imaginary eigenvalues $\lambda^\pm = \pm i\nu$. Choosing $K_1 = K_2 = \mathbb{1}$ we see that the three independent fields $Q_{10}(x,t)$, $Q_{01}(x,t)$ and $Q_{11}(x,t)$ are purely imaginary while the polarization vector is real, $|n\rangle^* = |n\rangle$. After introducing new variables

$$Q_{10}(x,t) = i\mathbf{q}_{10}(x,t), \qquad Q_{01}(x,t) = i\mathbf{q}_{01}(x,t), \qquad Q_{11}(x,t) = i\mathbf{q}_{11}(x,t),$$

we derive a real 3-wave system for three real-valued fields

$$\begin{aligned} &(J_1 - J_2)\mathbf{q}_{10,t} - (I_1 - I_2)\mathbf{q}_{10,x} + 3k\mathbf{q}_{11}\mathbf{q}_{01} = 0, \\ &(J_1 - J_3)\mathbf{q}_{11,t} - (I_1 - I_3)\mathbf{q}_{11,x} - 3k\mathbf{q}_{10}\mathbf{q}_{01} = 0, \\ &(J_2 - J_3)\mathbf{q}_{01,t} - (I_2 - I_3)\mathbf{q}_{01,x} + 3k\mathbf{q}_{10}\mathbf{q}_{11} = 0. \end{aligned} \tag{89}$$

Since the dressing factor must satisfy the conditions

$$(u^\dagger(x,\lambda^*))^{-1} = u(x,\lambda), \tag{90}$$

$$(u^T(x,-\lambda))^{-1} = u(x,\lambda), \tag{91}$$

the projector P is real valued. In this case the discrete eigenvalues are purely imaginary, i.e. $\lambda^\pm = \pm i\nu$. The one-soliton solution is

$$\mathbf{q}_{kl}^{1s}(x) = -2\nu P_{kl}(x), \qquad P = \frac{|n\rangle\langle n|}{\langle n|n\rangle}, \qquad k \neq l.$$

Taking into account that $|n\rangle = e^{\nu J x}|n_0\rangle$ we derive explicitly the following result:

$$\begin{aligned} \mathbf{q}_{10}(x,t) &= -\frac{2\nu\, e^{\nu(z_1+z_2)} n_{0,1} n_{0,2}}{e^{2\nu z_1} n_{0,1}^2 + e^{2\nu z_2} n_{0,2}^2 + e^{-2\nu(z_1+z_2)} n_{0,3}^2}, \\ \mathbf{q}_{11}(x,t) &= -\frac{2\nu\, e^{-\nu z_2} n_{0,1} n_{0,3}}{e^{2\nu z_1} n_{0,1}^2 + e^{2\nu z_2} n_{0,2}^2 + e^{-2\nu(z_1+z_2)} n_{0,3}^2}, \\ \mathbf{q}_{01}(x,t) &= -\frac{2\nu\, e^{-\nu z_1} n_{0,2} n_{0,3}}{e^{2\nu z_1} n_{0,1}^2 + e^{2\nu z_2} n_{0,2}^2 + e^{-2\nu(z_1+z_2)} n_{0,3}^2}. \end{aligned} \tag{92}$$

In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case there exists another type of soliton solutions—these obtained by using a dressing factor of the form

$$u(x,\lambda) = \mathbb{1} + (c(\lambda)-1)A(x) + \left(\frac{1}{c(-\lambda)} - 1\right) B(x), \qquad c(\lambda) = \frac{\lambda-\lambda^+}{\lambda-\lambda^-}, \tag{93}$$

while its inverse reads

$$u^{-1}(x,\lambda) = \mathbb{1} + \left(\frac{1}{c(\lambda)} - 1\right) C(x) + (c(-\lambda)-1)D(x). \tag{94}$$

These solutions are associated with four discrete eigenvalues of the scattering operator L: $\pm\lambda^{\pm}$. In this sense they may be called quadruplet solitons unlike the solutions (92) which being associated with two eigenvalues $\pm \mathrm{i}\nu$ represent doublet solitons. The dressing factor (93) is $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant if the conditions hold true

$$K_1(u^{\dagger}(x,\lambda^*))^{-1}K_1^{-1} = u(x,\lambda), \tag{95}$$

$$K_2(u^{T}(x,-\lambda))^{-1}K_2^{-1} = u(x,\lambda). \tag{96}$$

Hence the following relations hold:

$$B = K_1K_2A^*K_2^{-1}K_1^{-1}, \qquad C = K_1A^{\dagger}K_1^{-1}, \qquad D = K_2A^TK_2^{-1}, \tag{97}$$

as well as $\lambda^- = (\lambda^+)^* = \mu - \mathrm{i}\nu$.

The matrix-valued function A admits the decomposition

$$A(x) = X(x)F^T(x). \tag{98}$$

By using the equality $uu^{-1} = \mathbb{1}$ one can prove that the factor $X(x)$ can be expressed by $F(x)$ as follows:

$$X = \frac{1}{a^2-|b|^2}(aK_1F^* - b^*K_2F), \tag{99}$$

where

$$a = F^{\dagger}K_1F, \qquad b = -\frac{\mathrm{i}\nu F^TK_2F}{\mu-\mathrm{i}\nu}, \qquad F^T(x) = F_0^T(\chi_0^-(x,\lambda^-))^{-1}. \tag{100}$$

To find the one-soliton solution we take the limit $\lambda \to \infty$ in (23) and put $q_0 \equiv 0$. Thus we obtain the following formula:

$$Q_{jk}^{1s} = (\lambda^- - \lambda^+)(A + K_1K_2A^*K_2K_1)_{jk}, \qquad j \neq k. \tag{101}$$

Let $K_1 = K_2 = \mathbb{1}$. Then $Q^* = -Q$ and using the above notation we have for the one-soliton solution:

$$\mathbf{q}_{10} = -\frac{4\nu}{\Delta} f_{12}\sum_{k=1}^{3} f_{kk}\left\{\cos(\phi_1-\phi_2) - \frac{\nu\cos(2\phi_k-\phi_1-\phi_2+\phi_0)}{\sqrt{\mu^2+\nu^2}}\right\}, \tag{102}$$

$$\mathbf{q}_{11} = -\frac{4\nu}{\Delta} f_{13}\sum_{k=1}^{3} f_{kk}\left\{\cos(\phi_1-\phi_3) - \frac{\nu\cos(2\phi_k-\phi_1-\phi_3+\phi_0)}{\sqrt{\mu^2+\nu^2}}\right\}, \tag{103}$$

$$\mathbf{q}_{01} = -\frac{4\nu}{\Delta} f_{23}\sum_{k=1}^{3} f_{kk}\left\{\cos(\phi_2-\phi_3) - \frac{\nu\cos(2\phi_k-\phi_2-\phi_3+\phi_0)}{\sqrt{\mu^2+\nu^2}}\right\},$$

$$\Delta = a^2 - |b|^2 = \frac{1}{\mu^2+\nu^2}\left\{\mu^2\sum_{k=1}^{3} f_{kk}^2 + 2\sum_{k<p} f_{kp}^2[\mu^2 + 2\nu^2\sin^2(\phi_k-\phi_p)]\right\}, \tag{104}$$

$$f_{jk} = |F_{0,j}F_{0,k}|\,\mathrm{e}^{\nu(z_j+z_k)}, \qquad \phi_k(x,t) = \mu z_k + \delta_k, \qquad \delta_k = \arg F_{0,k},$$

where $\lambda^+ = \mathrm{i}\sqrt{\mu^2+\nu^2}\,\mathrm{e}^{\mathrm{i}\phi_0}$.

3.2. N-wave systems related to $sl(n)$, $n > 3$

For the sake of simplicity and clarity below, most of our discussions will be restricted to the case $n = 5$ and rank-1 projectors P. They also could easily be reformulated for any other chosen value of n and for higher rank projectors. The corresponding Lax operator $L(\lambda)$ which is a particular case of (3) with

$$L \equiv \mathrm{i}\partial_x + U(x,t,\lambda) = \mathrm{i}\partial_x + [J, Q(x,t)] - \lambda J,$$

$$J = \mathrm{diag}(J_1, J_2, J_3, J_4, J_5), \qquad Q(x,t) = \begin{pmatrix} 0 & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & 0 & Q_{23} & Q_{24} & Q_{25} \\ Q_{31} & Q_{32} & 0 & Q_{34} & Q_{35} \\ Q_{41} & Q_{42} & Q_{43} & 0 & Q_{45} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & 0 \end{pmatrix}.$$

Furthermore, for definiteness we will assume that

$$\mathrm{tr}\, J = 0, \qquad J_1 > J_2 > J_3 > 0, \qquad 0 > J_4 > J_5. \tag{106}$$

The M-operator in the Lax representation for the N-wave equation (10) is given by

$$M\psi(x,t,\lambda) \equiv \mathrm{i}\partial_t\psi + ([I, Q(x,t)] - \lambda I)\psi(x,t,\lambda) = -\lambda\psi(x,t,\lambda)I, \tag{107}$$

where $I = \mathrm{diag}(I_1, \ldots, I_5)$ is a traceless matrix. The generic one-soliton solution can be derived by using (32),

$$q(x) = \lim_{\lambda\to\infty} \lambda(J - u(x,\lambda)Ju^{-1}(x,\lambda)) = -(\lambda^+ - \lambda^-)[J, P(x)], \tag{108}$$

with a generic projector P whose rank s can be bigger than 1,

$$\begin{aligned} &P(x,t) = \sum_{a,b=1}^{s} |n_a(x,t)\rangle M^{-1}_{ab}\langle n^{\dagger}_b(x,t)|, \qquad M_{ab}(x,t) = \langle n^{\dagger}_b(x,t)|n_a(x,t)\rangle, \\ &|n_a(x,t)\rangle = \chi_0^+(x,t,\lambda^+)|n_{0,a}\rangle, \qquad \langle n_{0,a}|S_0|n_{0,b}\rangle = 0. \end{aligned} \tag{109}$$

Note that the set of s linearly independent polarization vectors $|n_k\rangle$ determine the corresponding eigensubspace of L. Such subspace can be defined either as the image of P or as the kernel of the projector $\widetilde{P} = \mathbb{1} - P$ which is defined by a complimentary set of $n - s$ polarization vectors. Therefore studying $sl(n)$-type Zakharov–Shabat systems it is enough to analyze projectors of rank $s \leqslant [n/2]$, where $[n/2]$ is the entire part of $n/2$. Thus for $n = 3$ it is enough to study rank-1 projectors, while for $n = 5$ one needs also rank-2 projectors.

For $s = 2$ we have two linearly independent polarization vectors $|n_a\rangle$, $a = 1, 2$ and from (109) we get

$$\begin{aligned} P(x,t) = \frac{1}{\det M}\big(&|n_1(x,t)\rangle M_{22}\langle n^{\dagger}_1(x,t)| - |n_2(x,t)\rangle M_{12}\langle n^{\dagger}_1(x,t)| \\ &- |n_1(x,t)\rangle M_{21}\langle n^{\dagger}_2(x,t)| + |n_2(x,t)\rangle M_{11}\langle n^{\dagger}_2(x,t)|\big), \\ \det M(x,t) = M_{11}M_{22} &- M_{12}M_{21}, \qquad M_{ab}(x,t) = \langle n^{\dagger}_a(x,t)|n_b(x,t)\rangle. \end{aligned} \tag{110}$$

Let first concentrate on rank-1 projectors, see (93) with

$$|n(x)\rangle = \chi_0^+(x,\lambda^+)|n_0\rangle, \qquad \langle m(x)| = \langle m_0|\hat{\chi}_0^-(x,\lambda^-). \tag{111}$$

The polarization vectors $|n_0\rangle$ and $\langle m_0|$ are constant 5-vectors. Thus these type of one-soliton solutions is parametrized by:

(i) The discrete eigenvalues $\lambda^{\pm} = \mu^{\pm} \pm \mathrm{i}\nu^{\pm}$; $\mu^{\pm}$ determine the soliton velocity, $\nu^{\pm}$ determine the amplitude.

(ii) The 'polarization' vectors $|n_0\rangle$, $\langle m_0|$ parametrize the internal degrees of freedom of the soliton. Note that $P(x)$ is invariant under the scaling of each of these vectors. Generically each 'polarization' has five components, one of which can be fixed, say to 1. So each 'polarization' is determined by four independent complex parameters.

We have several options that will lead to different types of solitons:

(1) generic case when all components of $|n_0\rangle$ are non-vanishing;

(2) several special subcases when one (or several) of these components vanish. The corresponding solitons will have different structures and properties.

For the generic choice of $|n_0\rangle$ one finds

$$\lim_{x\to\pm\infty} P(x,t) = P_\pm, \qquad P_+ = E_{11}, \qquad P_- = E_{nn}, \tag{112}$$

where the matrix E_{kj} has only one non-vanishing matrix element equal to 1 at position k, j, i.e. $(E_{kj})_{mp} = \delta_{km}\delta_{jp}$. Therefore both the limiting values $u_\pm(\lambda)$ and their inverse $\hat{u}_\pm(\lambda)$ are diagonal matrices

$$u_+(\lambda) = \mathrm{diag}(c(\lambda), 1, 1, \ldots, 1), \qquad u_-(\lambda) = \mathrm{diag}(1, 1, \ldots, 1, c(\lambda)). \tag{113}$$

From (25) for $n = 5$ we have

$$\begin{aligned} T_{1j}(\lambda) &= c(\lambda)(T_0)_{1j}(\lambda), \qquad j = 1, 2, 3, 4; \\ T_{j5}(\lambda) &= (T_0)_{j5}(\lambda)/c(\lambda), \qquad j = 2, 3, 4, 5; \end{aligned} \tag{114}$$

$$T_{ij}(\lambda) = (T_0)_{ij}(\lambda), \qquad \text{for all other values of } i, j. \tag{115}$$

This relation allows us to derive the interrelations between the Gauss factors of $T_0(\lambda)$ and $T(\lambda)$. In particular we find for the principal minors of $T(\lambda)$,

$$m_k^+(\lambda) = c(\lambda) m_{0,k}^+(\lambda), \qquad m_k^-(\lambda) = m_{0,k}^-(\lambda)/c(\lambda), \tag{116}$$

where $m_k^+(\lambda)$ (resp. $m_k^-(\lambda)$) are the upper (resp. lower) principal minors of $T(\lambda)$. Since $\chi_0^\pm(x,t,\lambda)$ are regular solutions of the RHP then $m_{0,k}^\pm(\lambda)$ have no zeroes at all, but (116) means *all* $m_k^\pm(\lambda)$ have a simple zero at $\lambda = \lambda^\pm$.

The generic one-soliton solution then is obtained by taking $\chi^\pm(x,t,\lambda) = \exp(-\mathrm{i}\lambda J x)$. As a result we get

$$(P(x,t))_{ks} = \frac{1}{k(x,t)} n_{0,k} m_{0,s}\, \mathrm{e}^{-\mathrm{i}(\lambda^+ z_k - \lambda^- z_s)}, \tag{117}$$

$$k(x,t) = \sum_{p=1}^{n} n_{0,p} m_{0,p}\, \mathrm{e}^{-\mathrm{i}(\lambda^+ - \lambda^-) z_p(x,t)}, \tag{118}$$

$$z_p(x,t) = J_p x + I_p t, \qquad q_{ks}^{1\mathrm{s}} = -(\lambda^+ - \lambda^-)(P(x,t))_{ks}, \tag{119}$$

i.e. in all channels we have nontrivial waves. The number of internal degrees of freedom is $2(n-1) = 8$. Note that the denominator $k(x,t)$ is a linear combination of exponentials with complex arguments, so it could vanish for certain values of x, t. Thus the generic soliton (117) in this case is a singular solution.

Next we impose on $U(x,t,\lambda)$ the involution

$$K U^\dagger(x,t,\lambda^*) K^{-1} = U(x,t,\lambda), \qquad K = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n), \tag{120}$$

with $\epsilon_j = \pm 1$. More specifically this means that

$$K q^\dagger(x,t) K^{-1} = q(x,t), \qquad K u^\dagger(x,t,\lambda^*) K^{-1} = u^{-1}(x,t,\lambda), \tag{121}$$

and

$$\lambda^{+} = (\lambda^{-})^{*} = \mu + \mathrm{i}\nu, \qquad \langle m_0| = (K|n_0\rangle)^{\dagger}. \tag{122}$$

Thus only $|n_0\rangle$ is independent.

Then the one-soliton solution simplifies to

$$q_{ks}^{1s}(x,t) = -\frac{2\mathrm{i}\nu(J_k - J_s)}{k_{\mathrm{red}}(x,t)}\epsilon_s n_{0,k} n_{0,s}^{*}\,\mathrm{e}^{\nu(z_k+z_s)}\,\mathrm{e}^{-\mathrm{i}\mu(z_k-z_s)}, \tag{123}$$

$$k_{\mathrm{red}}(x,t) = \sum_{p=1}^{n}\epsilon_p|n_{0,p}|^2\,\mathrm{e}^{2\nu z_p(x,t)}. \tag{124}$$

The number of internal degrees of freedom now is $n-1=4$. If one or more of ϵ_j are different, then this reduced soliton may still have singularities. The singularities are absent only if all ϵ_j are equal.

The analysis of solitons obtained with rank-2 projectors should go along the same lines. It is lengthier than the one above and we may omit it. We just note that even with the canonical reduction with $K = \mathbb{1}$ in (121), one cannot guarantee that $\det M > 0$ for all x and t (see (110)) which means that one may encounter singular solitons.

3.3. *Typical $sl(2)$ solitons*

The finite-dimensional irreducible representations of the algebra $sl(2)$ are labeled by their spin J and have dimension $2J+1$; the spin J can take half-integer positive values. The simplest representation with $J=\frac{1}{2}$ is two dimensional and is known also as the typical representation of $sl(2)$. In this subsection, we will address first the simplest possibility when $sl(2)$ is embedded into $sl(5)$ via its typical representation; we will call such solitons typical $sl(2)$ solitons.

From now on we assume that the reduction (120) with $\epsilon_p = 1$ holds. Here $|n_{0,1}\rangle$ has only two non-vanishing components. We consider here three examples with $n=5$ and three different choices for the polarization vectors

$$\text{(a)}\quad |n_0\rangle = \begin{pmatrix} n_{0,1}\\ 0\\ 0\\ 0\\ n_{0,5}\end{pmatrix};\qquad \text{(b)}\quad |n_0\rangle = \begin{pmatrix} 0\\ n_{0,2}\\ 0\\ n_{0,4}\\ 0\end{pmatrix};\qquad \text{(c)}\quad |n_0\rangle = \begin{pmatrix} n_{0,1}\\ n_{0,2}\\ 0\\ 0\\ 0\end{pmatrix}. \tag{125}$$

In all these cases the corresponding one-soliton solutions $q(x,t)$ are given by similar analytic expressions, each having only two non-vanishing matrix elements

$$q_{jk}(x,t) = (q_{jk}(x,t))^{*} = -\frac{\mathrm{i}\nu(J_j - J_k)\,\mathrm{e}^{\mathrm{i}(\arg(n_{0,j}) - \arg(n_{0,k}))}\,\mathrm{e}^{-\mathrm{i}\mu(J_j-J_k)(x+w_{jk}t)}}{\cosh[\nu(J_j-J_k)(x+w_{jk}t) + \ln|n_{0,j}| - \ln|n_{0,k}|]}, \tag{126}$$

where we recall that $w_{jk} = (I_j - I_k)/(J_j - J_k)$, $j<k$. For case (a) we have $j=1, k=5$; in case (b): $j=2, k=4$ and in case (c) $j=1$ and $k=2$.

The $sl(2)$ soliton is very much like the NLS soliton (apart from the t-dependence); the NLS soliton has only one internal degree of freedom.

The different choices for the polarization vector result in different asymptotics for the projector $P_1(x)$,

$$\begin{aligned}
&\text{(a)}\quad \lim_{x\to\infty}P(x) = E_{11}, \qquad \lim_{x\to-\infty}P(x) = E_{55},\\
&\text{(b)}\quad \lim_{x\to\infty}P(x) = E_{22}, \qquad \lim_{x\to-\infty}P(x) = E_{44},\\
&\text{(c)}\quad \lim_{x\to\infty}P(x) = E_{11}, \qquad \lim_{x\to-\infty}P(x) = E_{22}.
\end{aligned} \tag{127}$$

In case (a) the results for the limits of $P(x)$ and for $u_{\pm}(\lambda)$ are the same as for the generic case, see (112) and (113). As a consequence, such $sl(2)$ solitons require the vanishing of *all* Evans functions $m_k^{\pm}(\lambda)$ for $\lambda = \lambda^{\pm}$, see (116).

In case (b) from (25) and from the appendices we get that such $sl(2)$ soliton provides for the vanishing of $m_2^{\pm}(\lambda)$ and $m_3^{\pm}(\lambda)$,

$$\begin{aligned} m_2^+(\lambda) &= c(\lambda) m_{0,2}^+(\lambda), & m_3^+(\lambda) &= c(\lambda) m_{0,3}^+(\lambda), \\ m_2^-(\lambda) &= m_{0,2}^-(\lambda)/c(\lambda), & m_3^-(\lambda) &= m_{0,3}^-(\lambda)/c(\lambda), \end{aligned} \tag{128}$$

whereas $m_1^{\pm}(\lambda) = m_{0,1}^{\pm}(\lambda)$ and $m_4^{\pm}(\lambda) = m_{0,4}^{\pm}(\lambda)$ remain regular and do not have zeros at $\lambda = \lambda^{\pm}$.

Likewise in case (c) we get that only $m_1^+(\lambda)$ and $m_4^-(\lambda)$ acquire zeroes,

$$m_1^+(\lambda) = c(\lambda) m_{0,1}^+(\lambda), \qquad m_4^-(\lambda) = m_{0,4}^-(\lambda)/c(\lambda), \tag{129}$$

and all the other Evans functions $m_j^+(\lambda)$ with $j = 2, 3, 4$, and $m_p^-(\lambda)$ with $p = 1, 2, 3$ do not have zeroes.

3.4. Higher spin J sl(2) solitons

Here we provide some examples of $sl(2)$ soliton solutions which are embedded into $sl(5)$ via a higher $(2J+1)$-dimensional representation of $sl(2)$ which we call spin J $sl(2)$ solitons. Obviously for $\mathfrak{g} \simeq sl(5)$ the 'spin' of the solitons can take values $J = 1, 3/2$ and 2.

It is well known that spin J representation can be constructed using the completely symmetrized tensor products of the typical 2×2 representation. The details of their derivations are given in appendix B. Here we briefly formulate the results.

Before starting with to calculate symmetrized tensor products of the dressing factor we have to make a slight modification so that it takes values in the group $SL(2)$. Indeed, as it is given by (27) and (65) one easily checks that $\det u(x,\lambda) = c(\lambda)$. Therefore we multiply it by the constant $1/\sqrt{c(\lambda)}$ factor so that its determinant equals 1. Thus we can rewrite the dressing factor for the typical $sl(2)$ soliton in the form

$$u(x,\lambda) = \sqrt{c(\lambda)}\, P(x) + \frac{1}{\sqrt{c(\lambda)}} \bar{P}(x), \tag{130}$$

where $\bar{P} = \mathbb{1} - P$; in terms of the polarization vectors they take the form

$$P = \frac{1}{n_1 m_1 + n_2 m_2} \begin{pmatrix} n_1 m_1 & n_1 m_2 \\ n_2 m_1 & n_2 m_2 \end{pmatrix}, \qquad \bar{P} = \frac{1}{n_1 m_1 + n_2 m_2} \begin{pmatrix} n_2 m_2 & -n_1 m_2 \\ -n_2 m_1 & n_1 m_1 \end{pmatrix}. \tag{131}$$

After some calculations we get the following results for the higher spin dressing factors:

$$\begin{aligned} U^{(3)} &\equiv u \odot u = c(\lambda) \pi_1^{(3)} + \pi_0^{(3)} + \frac{1}{c(\lambda)} \pi_{-1}^{(3)}, \\ U^{(4)} &\equiv u \odot u \odot u = \sum_{l=-3/2}^{3/2} \pi_l^{(4)} c^l(\lambda), \\ U^{(5)} &\equiv u \odot u \odot u \odot u = \sum_{l=-2}^{2} \pi_l^{(5)} c^l(\lambda). \end{aligned} \tag{132}$$

By $u \odot u$, $u \odot u \odot u$, etc we have denoted the completely symmetrized part of the corresponding tensor powers of the dressing factor $u(x,\lambda)$, the superscript k of $U^{(k)}$ denotes the dimension $2J+1$ of the representation. By $\pi_a^{(k)}$ we have denoted mutually orthogonal rank-1 projectors

$$\pi_a^{(k)} = \frac{\left|N_a^{(k)}\right\rangle\left\langle M_a^{(k)}\right|}{\left\langle M_a^{(k)} \middle| N_a^{(k)}\right\rangle}, \qquad \pi_a^{(k)} \pi_b^{(k)} = \delta_{ab} \pi_a^{(k)}. \tag{133}$$

The explicit expressions for the vectors $\left|N_a^{(k)}\right\rangle$ and $\left\langle M_a^{(k)}\right|$ in terms of n_i and m_j are given in appendix A. Here we just mention that

$$\left\langle M_a^{(k)}\right| = \left\langle N_a^{(k)}\right| S_0^{(k)}, \tag{134}$$

where the matrices $S_0^{(k)}$

$$S_0^{(k)} = \begin{cases} \displaystyle\sum_{s=0}^{k} (-1)^{k+1} E_{s,k+1-s}^{(k)} & \text{for} \quad k = 2p+1 \\ \displaystyle\sum_{s=0}^{(k-1)/2} (-1)^{k+1} \left(E_{s,k+1-s}^{(k)} - E_{k+1-s,s}^{(k)}\right) & \text{for} \quad k = 2p \end{cases} \tag{135}$$

are the ones used to define the orthogonal algebras $so(k)$.

All these properties of the projectors $\pi_a^{(k)}$ allow us to rewrite the dressing factors in the form

$$U^{(k)}(x,\lambda) = \exp\left(\ln c(\lambda) \sum_{s=1}^{(k-1)/2} \left(\pi_s^{(k)} - \pi_{-s}^{(k)}\right) + \pi_0^{(k)}\right), \tag{136}$$

for odd values of k and

$$U^{(k)}(x,\lambda) = \exp\left(\ln c(\lambda) \sum_{s=1/2}^{(k-1)/2} \left(\pi_s^{(k)} - \pi_{-s}^{(k)}\right)\right) \tag{137}$$

for even values of k. It is important to note that due to (134) the differences $\pi_s^{(k)} - \pi_{-s}^{(k)} \in so(k)$ and also $\pi_0^{(k)} \in so(k)$.

Let us now outline how one can construct spin 3/2 soliton of the N-wave equations related to $sl(5)$. First we have to embed the dressing factor $U^{(4)}$ which is a 4×4 matrix into a 5×5 dressing factor. This can be done in several inequivalent ways, which reflects the fact that the group $SO(4)$ can be embedded into the group $SL(5)$ in different ways. As first of them we choose the following one. First we extend the four-component vectors $\left|N_a^{(k)}\right\rangle$ into five-component ones $\left|\mathbf{N}_a^{(k)}\right\rangle = \left(\left\langle N_a^{(k)}\right|, 0\right)^T$. We will need also the vector $|\mathbf{e}_5\rangle = (0,0,0,0,1)^T$. Then we can construct the 5×5 dressing factor

$$\mathbf{U}^{(4)} = \pi_{-3/2}^{(4)} c^{-3/2}(\lambda) + \pi_{-1/2}^{(4)} c^{-1/2}(\lambda) + \pi_{1/2}^{(4)} c^{1/2}(\lambda) + \pi_{3/2}^{(4)} c^{3/2}(\lambda), \tag{138}$$

where

$$\pi_a^{(4)} = \frac{\left|\mathbf{N}_a^{(4)}\right\rangle\left\langle \mathbf{M}_a^{(4)}\right|}{\left\langle \mathbf{M}_a^{(4)}\middle|\mathbf{N}_a^{(4)}\right\rangle}, \qquad \pi_0^{(4)} = \frac{\left|\mathbf{N}_0^{(4)}\right\rangle\left\langle \mathbf{M}_0^{(4)}\right|}{\left\langle \mathbf{M}_0^{(5)}\middle|\mathbf{N}_0^{(4)}\right\rangle} + |\mathbf{e}_5\rangle\langle\mathbf{e}_5|. \tag{139}$$

In order to calculate the corresponding soliton solution it remains to insert $\mathbf{U}^{(5)}$ as $u(x,\lambda)$ into (108). For $Q_{(0)} = 0$ the result will be

$$Q_{3/2} = \begin{pmatrix} 0 & \sqrt{3}q^+ & 0 & 0 & 0 \\ \sqrt{3}q^- & 0 & 2q^+ & 0 & 0 \\ 0 & 2q^- & 0 & \sqrt{3}q^+ & 0 \\ 0 & 0 & \sqrt{3}q^- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{140}$$

where the functions $q^{\pm}(x,t)$ are given by (59).

It is natural to analyze the structure of the discrete eigenvalues $\lambda^{\pm}$ corresponding to these types of soliton solutions. The discrete eigenvalues of L are the zeroes of the principal minors

$m_k^{\pm}(\lambda)$ of the scattering matrix $T(\lambda)$. To this end we have to calculate the diagonal factors $D^{\pm}(\lambda)$ of the Gauss decomposition of $T(\lambda)$. The result is

$$\begin{aligned} m_1^+(\lambda) &= c^3(\lambda) m_{0;1}^+(\lambda), \qquad & m_2^+(\lambda) &= c^4(\lambda) m_{0;2}^+(\lambda), \\ m_3^+(\lambda) &= c^3(\lambda) m_{0;3}^+(\lambda), \qquad & m_4^+(\lambda) &= m_{0;4}^+(\lambda). \end{aligned} \tag{141}$$

Comparing with (129) we conclude that going to higher representations of $sl(2)$ leads to multiple zeroes of some of the principle minors of $T(\lambda)$. Therefore the resolvent of L will have poles of higher order at $\lambda^{\pm}$, though the residues at this points will be expressed by the same polarization vectors as for the $\frac{1}{2}$-spin solitons. One consequence of this construction is that the higher spin $sl(2)$ solitons have the two degrees of freedom as the standard (spin $\frac{1}{2}$) solitons.

3.5. Typical $sl(3)$-solitons

Here $|n_0\rangle$ has three non-vanishing components. We consider three examples of such polarization vectors

$$\text{(a)}\quad |n_0\rangle = \begin{pmatrix} n_{0,1} \\ 0 \\ n_{0,3} \\ 0 \\ n_{0,5} \end{pmatrix}, \qquad \text{(b)}\quad |n_0\rangle = \begin{pmatrix} 0 \\ n_{0,2} \\ n_{0,3} \\ n_{0,4} \\ 0 \end{pmatrix}, \qquad \text{(c)}\quad |n_0\rangle = \begin{pmatrix} n_{0,1} \\ n_{0,2} \\ n_{0,3} \\ 0 \\ 0 \end{pmatrix}. \tag{142}$$

Therefore the $sl(3)$-solitons have two internal degrees of freedom.

The asymptotics of the projector $P(x)$ read as follows:

$$\begin{aligned} &\text{(a)} \quad \lim_{x\to\infty} P(x) = E_{11}, \qquad && \lim_{x\to-\infty} P(x) = E_{55}, \\ &\text{(b)} \quad \lim_{x\to\infty} P(x) = E_{22}, \qquad && \lim_{x\to-\infty} P(x) = E_{44}, \\ &\text{(c)} \quad \lim_{x\to\infty} P(x) = E_{11}, \qquad && \lim_{x\to-\infty} P(x) = E_{33}. \end{aligned} \tag{143}$$

Note that cases (a) and (b) in (143) coincide with the corresponding cases in (125). Therefore the set of Evans functions that acquire zeroes will be the same as for the corresponding $sl(2)$ solitons. In case (c) of (143) we have

$$\begin{aligned} m_1^+(\lambda) &= c(\lambda) m_{0,1}^+(\lambda), \qquad & m_2^+(\lambda) &= c(\lambda) m_{0,2}^+(\lambda), \\ m_4^-(\lambda) &= m_{0,4}^-(\lambda)/c\lambda), \qquad & m_3^-(\lambda) &= m_{0,3}^-(\lambda)/c(\lambda), \end{aligned} \tag{144}$$

whereas the remaining Evans functions $m_j^+(\lambda)$ with $j = 3, 4$, and $m_p^-(\lambda)$ with $p = 1, 2$ remain regular.

In case (a) the corresponding one-soliton solutions acquire the form

$$\text{(a)}\quad q^{1s} = \begin{pmatrix} 0 & 0 & q_{13} & 0 & q_{15} \\ 0 & 0 & 0 & 0 & 0 \\ q_{13}^* & 0 & 0 & 0 & q_{35} \\ 0 & 0 & 0 & 0 & 0 \\ q_{15}^* & 0 & q_{35}^* & 0 & 0 \end{pmatrix}, \qquad \text{(b)}\quad q^{1s} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{23} & q_{24} & 0 \\ 0 & q_{23}^* & 0 & q_{34} & 0 \\ 0 & q_{24}^* & q_{34}^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{(c)}\quad q^{1s} = \begin{pmatrix} 0 & q_{12} & q_{23} & 0 & 0 \\ q_{12}^* & 0 & q_{23} & 0 & 0 \\ q_{13}^* & q_{23}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{145}$$

where the matrix elements $q_{ks}(x,t)$ are given by

$$q_{ks}(x,t) = -\frac{\mathrm{i}\nu(J_k - J_s)\epsilon_s\, \mathrm{e}^{\nu(\widetilde{J}_k+\widetilde{J}_s)(x+\widetilde{v}_{ks}t)} n_{0,k} n_{0,s}^{*}\, \mathrm{e}^{-\mathrm{i}\mu(J_k-J_s)(x+w_{15}t)}}{|n_{0,1}|^2\, \mathrm{e}^{2\nu(\widetilde{J}_1 x+\widetilde{I}_1 t)} + |n_{0,3}|^2\, \mathrm{e}^{2\nu(\widetilde{J}_3 x+\widetilde{I}_3 t)} + |n_{0,5}|^2\, \mathrm{e}^{2\nu(\widetilde{J}_5 x+\widetilde{I}_5 t)}}, \tag{146}$$

and

$$\widetilde{J}_k = J_k - (J_1 + J_3 + J_5)/3, \qquad \widetilde{I}_k = I_k - (I_1 + I_3 + I_5)/3, \qquad \widetilde{v}_{ks} = \frac{\widetilde{J}_k + \widetilde{J}_s}{\widetilde{I}_k + \widetilde{I}_s}. \tag{147}$$

This soliton has two internal degrees of freedom and is regular.

Obviously it is by now clear how one can write more complicated solitons like $sl(4)$ which would be characterized by the polarization vectors of the form

$$\text{(a)}\quad |n_0\rangle = \begin{pmatrix} n_{0,1} \\ n_{0,2} \\ n_{0,3} \\ n_{0,4} \\ 0 \end{pmatrix}, \qquad \text{(b)}\quad |n_0\rangle = \begin{pmatrix} n_{0,1} \\ n_{0,2} \\ n_{0,3} \\ 0 \\ n_{0,5} \end{pmatrix}, \qquad \ldots. \tag{148}$$

The $sl(4)$-solitons will have three internal degrees of freedom.

We note here that due to our choice of J in (106), $sl(4)$-solitons cannot give rise to generalized eigenfunctions.

Of course one could try to embed $sl(3)$ into g using one of its higher dimensional representations. The algebra $sl(3)$ has a second three-dimensional representation which is obtained related to the typical one used above by an exterior automorphism. The corresponding soliton solutions are obtained from those demonstrated above by a re-parameterization of the polarization vectors. The next irreducible representation of $sl(3)$ which is the adjoint representation, has dimension 8, so it cannot be embedded into $sl(5)$. Of course, considering algebras g of rank high enough it is possible to embed subalgebras g_0 using their non-typical representations.

4. Eigenfunctions and eigensubspaces

The structure of these eigensubspaces and the corresponding solitons becomes more complicated with the growth of n.

In what follows we start with the generic case and split the 'polarization' vector into two parts

$$|n_0\rangle = |p_0\rangle + |d_0\rangle; \qquad |p_0\rangle = \begin{pmatrix} n_{0,1} \\ n_{0,2} \\ n_{0,3} \\ 0 \\ 0 \end{pmatrix}, \qquad |d_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ n_{0,4} \\ n_{0,5} \end{pmatrix}, \tag{149}$$

and therefore

$$|n\rangle = |p\rangle + |d\rangle, \qquad |p\rangle = \chi_0^+(x,\lambda^+)|p_0\rangle, \qquad |d\rangle = \chi_0^+(x,\lambda^+)|d_0\rangle. \tag{150}$$

This splitting is compatible with (106) and has the advantage: if $\chi_0^+(x,\lambda^+) = \exp(-\mathrm{i}\lambda^+ Jx)$ then $|p\rangle$ increases exponentially for $x \to \infty$ and decreases exponentially for $x \to -\infty$; $|d\rangle$ decreases exponentially for $x \to \infty$ and increases exponentially for $x \to -\infty$, see also lemma 1.

What we will prove below is that one can take a special linear combination of the columns of $\chi_0^+(x,\lambda^+)$ which decreases exponentially for both $x\to\infty$ and $x\to-\infty$. Doing this we will use the fact that

$$\chi^+(x,\lambda^+)|n_0\rangle \equiv (\mathbb{1}-P(x))\chi^+(x,\lambda^+)|n_0\rangle = (\mathbb{1}-P(x))|n(x)\rangle = 0. \tag{151}$$

Lemma 1. *The eigenfunctions of L provided by*

$$\boldsymbol{f}^+(x) = \chi^+(x,\lambda^+)|p_0\rangle = -\chi^+(x,\lambda^+)|d_0\rangle, \tag{152}$$

decrease exponentially for both $x\to\infty$ and $x\to-\infty$.

Proof. From (151) and (149) there follows that both expressions for $\boldsymbol{f}^+(x,t)$ coincide, so we can use each of them to our advantage, see (152). We will use also the fact that $\mathbb{1}-P(x)$ is a bounded function of x.

We start with

$$\lim_{x\to\infty}\boldsymbol{f}^+(x) = \lim_{x\to\infty}\chi^{\prime,+}(x,\lambda^+)|d_0\rangle = (\mathbb{1}-P_+)\lim_{x\to\infty}\mathrm{e}^{-\mathrm{i}\lambda^+Jx}\mathbb{T}^-(\lambda^+)|d_0\rangle, \tag{153}$$

where $\mathbb{T}^-(\lambda^+)$ is the lower triangular matrix introduced in (17). If the potential is on finite support or is reflectionless then $\mathbb{T}^-(\lambda)$ is rational function well defined for $\lambda=\lambda^+$. If the potential is generic then $\mathbb{T}^-(\lambda)$ does not allow analytic continuation off the real axis. Nevertheless $\mathbb{T}^-(\lambda^+)$ can be understood as lower triangular constant matrix (generalizing the constant C_0^+ of the NLS case). Being lower triangular $\mathbb{T}^-(\lambda^+)$ maps $|d_0\rangle$ onto $|d_0'\rangle=\mathbb{T}_0^-(\lambda^+)|d_0\rangle$ which is again of the form (149), i.e. its first three components vanish. Therefore

$$\lim_{x\to\infty}\mathrm{e}^{\nu ax}\boldsymbol{f}^+(x) = \lim_{x\to\infty}(\mathbb{1}-P_+)\,\mathrm{e}^{\nu ax}\begin{pmatrix}0\\0\\0\\ \mathrm{e}^{-\mathrm{i}\lambda^+J_4x}n'_{0,4}\\ \mathrm{e}^{-\mathrm{i}\lambda^+J_5x}n'_{0,5}\end{pmatrix}=0, \tag{154}$$

for any constant $a>0$ such that $a+J_4<0$.

Likewise we can calculate the limit for $x\to-\infty$

$$\lim_{x\to-\infty}\boldsymbol{f}^+(x) = -\lim_{x\to-\infty}\chi^{\prime,+}\big(x,\lambda_1^+\big)|p_0\rangle = -(\mathbb{1}-P_+)\lim_{x\to\infty}\mathrm{e}^{-\mathrm{i}\lambda^+Jx}\mathbb{S}^+(\lambda^+)|p_0\rangle. \tag{155}$$

The upper triangular matrix $\mathbb{S}^+(\lambda^+)$ is treated analogously as $\mathbb{T}^-(\lambda^+)$. In the generic case it is just an upper triangular constant matrix which maps $|p_0\rangle$ onto $|p_0'\rangle=\mathbb{S}^+(\lambda^+)|p_0\rangle$ whose last two components vanish. Therefore

$$\lim_{x\to-\infty}\mathrm{e}^{\nu bx}\boldsymbol{f}^+(x) = \lim_{x\to-\infty}\mathrm{e}^{\nu bx}(\mathbb{1}-P_-)\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\lambda^+J_1x}n'_{0,1}\\ \mathrm{e}^{-\mathrm{i}\lambda^+J_2x}n'_{0,2}\\ \mathrm{e}^{-\mathrm{i}\lambda^+J_3x}n'_{0,3}\\0\\0\end{pmatrix}=0, \tag{156}$$

for any constant $b<0$ such that $J_3+b>0$.

The lemma is proved. □

For the choices (a) and (b) of $|n_0\rangle$ in (125) we define the square integrable discrete eigenfunctions using the splitting (149) and (152).

Remark 4. The choice (c) for $|n_0\rangle$ does not allow for the splitting (149). In this case we can introduce only generalized discrete eigenfunctions, $\boldsymbol{f}_{\rm gen}(x,t)$, which are not square integrable. But upon multiplying by the exponential factor $\mathrm{e}^{-\nu c x}$ with $c = (J_1 + J_2)/2$, we can obtain square integrable functions $\boldsymbol{f}(x) = \boldsymbol{f}_{\rm gen}(x)\,\mathrm{e}^{-\nu c x}$. See also the discussion in the following subsection.

The generalized eigenfunctions come up in situations when the splitting (149) is not possible, i.e. when either $|p_0\rangle$ or $|d_0\rangle$ vanish. Let us construct the generalized eigenfunction for the polarization vector $|n_0\rangle$ of case (c) in (142). Let $(J_1 + J_2 + J_3)/3 = a'$; then $J_1' = J_1 - a'$, $J_2' = J_2 - a'$ and $J_3' = J_3 - a'$ are such that $J_1' > J_2' > J_3'$ and $J_1' + J_2' + J_3' = 0$. Let us assume for definiteness that $J_1' > J_2' > 0$ and $0 > J_3'$. Then we can split $|n_0\rangle$ into

$$|n_0\rangle = |p_0'\rangle + |d_0'\rangle, \qquad |p_0'\rangle = \begin{pmatrix} n_{0,1} \\ n_{0,2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |d_0'\rangle = \begin{pmatrix} 0 \\ 0 \\ n_{0,3} \\ 0 \\ 0 \end{pmatrix}, \tag{157}$$

and define

$$\boldsymbol{f}^{+,\prime}(x) = \chi^+(x,\lambda^+)|p_0'\rangle = -\chi^+(x,\lambda^+)|d_0'\rangle. \tag{158}$$

Obviously $\boldsymbol{f}^{+,\prime}(x)$ is an eigenfunction of the dressed operator L corresponding to the eigenvalue λ^+.

Then we can prove the following lemma.

Lemma 2. *The eigenfunction* $\boldsymbol{f}^{+,\prime}(x)$ *is such that* $\mathrm{e}^{\nu_1 a' x}\boldsymbol{f}^{+,\prime}(x)$ *decreases exponentially for both* $x \to \pm\infty$.

Proof. The proof is similar to the one of lemma 1 and we omit it. □

Since the polarization vector $|n_0\rangle$ in case (c) of (142) does not allow the splitting (149) the corresponding discrete eigenfunction will not be square integrable, so it will give rise to a generalized eigenfunction.

5. Effects of reductions on soliton solutions

In this section, we analyze how different kinds of reductions affect the classification of the soliton solutions to a nonlinear equation. This criterion is tightly connected with symmetries imposed on the auxiliary linear problem (the zero curvature condition). We shall consider in the next subsection types of solitons which differ from one another in the number of eigenvalues associated with them: doublet solitons associated with two purely imaginary eigenvalues $\lambda^\pm = \pm i\nu$ and quadruplet solitons associated with four eigenvalues situated symmetrically with respect to the real and the imaginary axis in $\mathbb{C}$. This is the case when a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction is in action. Such a type of reduction is compatible with the Lax representation of a NLEE to have a dispersion law obeying $f(-\lambda) = -f(\lambda)$. The N-wave equation is a typical example of NLEE to admit a variety of $\mathbb{Z}_2$ reductions [27] since $f_{N-\mathrm{W}}(\lambda) = -\lambda I$.

It is also possible to impose reductions that do not affect the spectral parameter. Thus, imposing such reduction on the generalized ZSS related to the exterior automorphism of $sl(n)$ we can effectively reduce it to the generalized ZSS system for its subalgebras $so(n)$ and to the symmetric space of **BD.I** type.

For further convenience we shall say that the matrix X belongs to the orthogonal algebra $so(n)$ if

$$X + S_0^{(n)} X^T S_0^{(n)} = 0, \qquad S_0^{(n)} S_0^{(n)} = \mathbb{1}, \tag{159}$$

where $S_0^{(n)}$ is defined by

$$S_0^{(n)} = \sum_{s=1}^{n+1} (-1)^{s+1} E_{s,n+1-s}^{(n)}, \tag{160}$$

for $n = 2r + 1$ and

$$S_0^{(n)} = \sum_{s=1}^{r} (-1)^{s+1} \left(E_{s,n+1-s}^{(n)} + E_{n+1-s,s}^{(n)} \right) \tag{161}$$

for $n = 2r$. With this definition of orthogonality the Cartan subalgebra generators are represented by diagonal matrices. By $E_{sp}^{(n)}$ above we mean $n \times n$ matrix whose matrix elements are $\left(E_{sp}^{(n)}\right)_{ij} = \delta_{si}\delta_{pj}$.

In order to get the ZSS related to the symmetric space of **BD.I** type we have to specify $J = \mathrm{diag}(1, 0, \ldots, 0, -1)$ and take $q(x, t) = [J, Q(x, t)]$ where $Q(x, t) \in so(n)$. In this case the NLEE with linear dispersion law become trivial. However, one can consider NLEE with quadratic and cubic dispersion laws which are multicomponent generalizations of NLS and mKdV equations respectively.

5.1. *N*-wave system related to $so(5)$

From now on we shall focus our attention on a N-wave equation related to the $so(5)$ algebra. The corresponding Lax pair is given by (3), (4) where

$$\begin{aligned} &U(x, t, \lambda) = [J, Q(x, t)] - \lambda J, \qquad &&V(x, t, \lambda) = [I, Q(x, t)] - \lambda I, \\ &J = \mathrm{diag}(J_1, J_2, 0, -J_2, -J_1), \qquad &&I = \mathrm{diag}(I_1, I_2, 0, -I_2, -I_1), \end{aligned} \tag{162}$$

and $Q(x, t)$ is a function taking values in $so(5)$. This algebra has two simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2$, and two more positive roots: $\alpha_1 + \alpha_2 = e_1$ and $\alpha_1 + 2\alpha_2 = e_1 + e_2 = \alpha_{\max}$. When they come as indices, e.g. in Q_α we will replace them by sequences of two integers: $\alpha \to kn$ if $\alpha = k\alpha_1 + n\alpha_2$. Moreover, we are going to use the auxiliary notation $\overline{kn} = -k\alpha_1 - n\alpha_2$. Thus the N-wave system itself consists of eight equations. Half of them read

$$\begin{aligned} &\mathrm{i}(J_1 - J_2)Q_{10,t}(x, t) - \mathrm{i}(I_1 - I_2)Q_{10,x}(x, t) + kQ_{11}(x, t)Q_{\overline{01}}(x, t) = 0, \\ &\mathrm{i}J_1 Q_{11,t}(x, t) - \mathrm{i}I_1 Q_{11,x}(x, t) - k(Q_{10}Q_{01} - Q_{12}Q_{\overline{01}})(x, t) = 0, \\ &\mathrm{i}(J_1 + J_2)Q_{12,t}(x, t) - \mathrm{i}(I_1 + I_2)Q_{12,x}(x, t) - kQ_{11}(x, t)Q_{01}(x, t) = 0, \\ &\mathrm{i}J_2 Q_{01,t}(x, t) - \mathrm{i}I_2 Q_{01,x}(x, t) + k(Q_{\overline{11}}Q_{12} + Q_{\overline{10}}Q_{11})(x, t) = 0, \end{aligned} \tag{163}$$

where $k := J_1 I_2 - J_2 I_1$ is a constant describing the wave interaction. The other four equations can be derived from those above by using the formal transformation $Q_{kn} \leftrightarrow Q_{\overline{kn}}$. One is able to integrate the system by applying the already discussed ideas—dressing method, etc. For that purpose we make use of the dressing factor (38). The one-soliton solution reads

$$\begin{aligned}
Q_{10}(x,t) &= \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}(\mathrm{e}^{-\mathrm{i}(\lambda^+ z_1 - \lambda^- z_2)} n_{0,1} m_{0,2} + \mathrm{e}^{\mathrm{i}(\lambda^+ z_2 - \lambda^- z_1)} n_{0,4} m_{0,5}),\\
Q_{11}(x,t) &= \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}(\mathrm{e}^{-\mathrm{i}\lambda^+ z_1} n_{0,1} m_{0,3} - \mathrm{e}^{-\mathrm{i}\lambda^- z_1} n_{0,3} m_{0,5}),\\
Q_{12}(x,t) &= \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}(\mathrm{e}^{-\mathrm{i}(\lambda^+ z_1 + \lambda^- z_2)} n_{0,1} m_{0,4} + \mathrm{e}^{-\mathrm{i}(\lambda^- z_1 + \lambda^+ z_2)} n_{0,2} m_{0,5}),\\
Q_{01}(x,t) &= \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}(\mathrm{e}^{-\mathrm{i}\lambda^+ z_2} n_{0,2} m_{0,3} + \mathrm{e}^{-\mathrm{i}\lambda^- z_2} n_{0,3} m_{0,4}),\\
\langle m|n\rangle &= \sum_{k=1}^{5} \mathrm{e}^{-\mathrm{i}(\lambda^+ - \lambda^-) z_k} n_{0,k} m_{0,k}, \qquad z_k = J_k x + I_k t, \qquad k = 1, 2.
\end{aligned} \tag{164}$$

The other four field can be formally constructed by doing the following transformation:

$$Q_{kn} \leftrightarrow Q_{\overline{kn}}, \qquad \mathrm{e}^{-\mathrm{i}\lambda^+ z_k} \leftrightarrow \mathrm{e}^{\mathrm{i}\lambda^- z_k}, \qquad n_{0,j} \leftrightarrow m_{0,j}.$$

For the typical $\mathbb{Z}_2$ reduction (120) of course we must choose $K_0 = \mathrm{diag}(\epsilon_1, \epsilon_2, 1, \epsilon_2, \epsilon_1) \in SO(5)$ where $\epsilon_i = \pm 1$. As a result J and I become real valued and $K_0 Q^\dagger K_0^{-1} = -Q$, i.e.,

$$Q_{\overline{10}} = -\epsilon_1\epsilon_2 Q_{10}^*, \qquad Q_{\overline{12}} = -\epsilon_1\epsilon_2 Q_{12}^*, \qquad Q_{\overline{11}} = -\epsilon_1 Q_{11}^*, \qquad Q_{01}^* = -\epsilon_2 Q_{01}. \tag{165}$$

The corresponding 4-wave system takes the form

$$\begin{aligned}
&\mathrm{i}(J_1 - J_2) Q_{10,t}(x,t) - \mathrm{i}(I_1 - I_2) Q_{10,x}(x,t) - k\epsilon_2 Q_{11}(x,t) Q_{01}^*(x,t) = 0,\\
&\mathrm{i}J_1 Q_{11,t}(x,t) - \mathrm{i}I_1 Q_{11,x}(x,t) - k(Q_{10}Q_{01} + \epsilon_2 Q_{12} Q_{01}^*)(x,t) = 0,\\
&\mathrm{i}(J_1 + J_2) Q_{12,t}(x,t) - \mathrm{i}(I_1 + I_2) Q_{12,x}(x,t) - k Q_{11}(x,t) Q_{01}(x,t) = 0,\\
&\mathrm{i}J_2 Q_{01,t}(x,t) - \mathrm{i}I_2 Q_{01,x}(x,t) - k\epsilon_1 (Q_{11}^* Q_{12} + \epsilon_2 Q_{10}^* Q_{11})(x,t) = 0.
\end{aligned} \tag{166}$$

The particular case $\epsilon_1 = \epsilon_2 = 1$ occurs in Raman scattering [28].

The corresponding one-soliton solution is obtained from (164) imposing $\lambda^+ = (\lambda^-)^* = \mu + \mathrm{i}\nu$ and $|m_0\rangle = K_0|n_0^*\rangle$. Here we just note that

$$\langle m|n\rangle = \epsilon_1(\mathrm{e}^{-2\nu z_1}|n_{0,1}|^2 + \mathrm{e}^{2\nu z_1}|n_{0,5}|^2) + \epsilon_2(\mathrm{e}^{-2\nu z_2}|n_{0,2}|^2 + \mathrm{e}^{2\nu z_2}|n_{0,4}|^2) + |n_{0,3}|^2. \tag{167}$$

Taking $\epsilon_1 = \epsilon_2 = 1$ we find that $\langle m|n\rangle$ is positive for all x and t. If $\epsilon_1\epsilon_2 = -1$ the product $\langle m|n\rangle$ may vanish for finite x and t, i.e. the corresponding soliton is singular.

5.2. $\mathbb{Z}_2$ *reduction related to Weyl group elements*

Here we consider several $\mathbb{Z}_2$ reductions related to Weyl group elements. The first one is of the type

$$K_1 U^*(\lambda^*) K_1^{-1} = -U(\lambda) \qquad \Rightarrow \qquad K_1 Q^* K_1^{-1} = Q, \qquad K_1 J^* K_1^{-1} = -J, \tag{168}$$

where K_1 corresponds to the Weyl reflection $S_{e_1 - e_2}$,

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{169}$$

This reduction leads to the requirements $J_2 = -J_1^*$, $I_2 = -I_1^*$ for J and I respectively, and the following ones for Q,

$$Q_{\overline{10}} = Q_{10}^*, \qquad Q_{01} = -Q_{11}^*, \qquad Q_{12}^* = -Q_{12}, \qquad Q_{\overline{01}} = -Q_{\overline{11}}^*, \qquad Q_{\overline{12}}^* = -Q_{\overline{12}}.$$

The $\mathbb{Z}_2$ reduced integrable system consists of the following five equations for three complex Q_{10}, Q_{11}, $Q_{\overline{11}}$ and two real fields $Q_{12} = \mathbf{i}\mathbf{q}_{12}$, $Q_{\overline{12}} = \mathbf{i}\mathbf{q}_{\overline{12}}$:

$$\begin{aligned}
&2J_{10}Q_{10,t}(x,t) - 2I_{1,0}Q_{10,x}(x,t) - k_0 Q_{11}(x,t)Q_{\overline{11}}^*(x,t) = 0,\\
&J_1 Q_{11,t}(x,t) - I_1 Q_{11,x}(x,t) + k_0(Q_{10}Q_{11}^* - \mathbf{i}\mathbf{q}_{12}Q_{\overline{11}}^*)(x,t) = 0,\\
&2J_{1,1}\mathbf{q}_{12,t}(x,t) - 2I_{1,1}\mathbf{q}_{12,x}(x,t) - k_0|Q_{11}(x,t)|^2 = 0,\\
&J_1 Q_{\overline{11},t}(x,t) - I_1 Q_{\overline{11},x}(x,t) + k_0(Q_{\overline{10}}Q_{\overline{11}}^* - \mathbf{i}\mathbf{q}_{\overline{12}}Q_{11}^*)(x,t) = 0,\\
&2J_{1,1}Q_{\overline{12},t}(x,t) - 2I_{1,1}Q_{\overline{12},x}(x,t) - k_0|Q_{\overline{11}}(x,t)|^2 = 0,
\end{aligned} \tag{170}$$

where $J_1 = J_{0,1} + \mathrm{i}J_{1,1}$, $I_1 = I_{0,1} + \mathrm{i}I_{1,1}$ and the interaction constant $k_0 = -\mathrm{i}k$ is real.

Next we consider a $\mathbb{Z}_2$ reduction of the type

$$K_2 U^\dagger(\lambda^*)K_2^{-1} = U(\lambda) \qquad \Rightarrow \qquad K_2 J^* K_2^{-1} = J, \qquad K_2 Q^\dagger K_2^{-1} = -Q,$$

with $K = S_{e_2}$, i.e.,

$$K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{171}$$

Therefore we have the following relations:

$$\begin{aligned}
&J_1^* = J_1, \qquad J_2^* = -J_2, \qquad Q_{\overline{10}} = -Q_{12}^*, \qquad Q_{\overline{12}} = -Q_{10}^*,\\
&Q_{\overline{11}} = Q_{11}^*, \qquad Q_{01}^* = Q_{01}, \qquad Q_{\overline{01}}^* = Q_{\overline{01}}.
\end{aligned}$$

Thus we derive the following 5-wave system:

$$\begin{aligned}
&(J_1 - J_2)Q_{10,t}(x,t) - (I_1 - I_2)Q_{10,x}(x,t) + k_0 Q_{11}(x,t)Q_{\overline{01}}(x,t) = 0,\\
&J_1 Q_{11,t}(x,t) - I_1 Q_{11,x}(x,t) - k_0(Q_{10}Q_{01} - Q_{12}Q_{\overline{01}})(x,t) = 0,\\
&(J_1 + J_2)Q_{12,t}(x,t) - (I_1 + I_2)Q_{12,x}(x,t) - k_0 Q_{11}(x,t)Q_{01}(x,t) = 0,\\
&J_2 Q_{01,t}(x,t) - I_2 Q_{01,x}(x,t) + k_0(Q_{11}^* Q_{12} - Q_{12}^* Q_{11})(x,t) = 0,\\
&J_2 Q_{\overline{01},t}(x,t) - I_2 Q_{\overline{01},x}(x,t) - k_0(Q_{11}Q_{10}^* - Q_{10}Q_{11}^*)(x,t) = 0,
\end{aligned} \tag{172}$$

where $k_0 = -\mathrm{i}k$ is real.

The last two reductions requested complex valued J and I. As a result the direct and inverse spectral problems for the corresponding Lax operator L become more complicated [26]. In particular, the continuous spectrum of L fills up a bunch of lines in the complex λ-plane intersecting at the origin. The construction of the corresponding fundamental analytic solutions and the dressing factors requires additional care and will be discussed elsewhere.

5.3. Reductions of MNLS-type systems related to $so(5)$

MNLS equations related to $so(n+2)$ algebras have a Lax representation of the following type:

$$L = \mathrm{i}\partial_x + q(x,t) - \lambda J, \tag{173}$$

$$M = \mathrm{i}\partial_t + V_0(x,t) + V_1(x,t)\lambda - \lambda^2 J, \tag{174}$$

$$V_1(x,t) = q(x,t), \qquad V_0(x,t) = \mathrm{i}\,\mathrm{ad}_J\,\partial_x q + \tfrac{1}{2}[\mathrm{ad}_J q, q]. \tag{175}$$

Here J is the element of the Cartan subalgebra of $so(n)$ dual to e_1 and

$$q := \begin{pmatrix} 0 & \mathbf{q}^T & 0 \\ \mathbf{p} & 0 & s_0\mathbf{q} \\ 0 & \mathbf{p}^T s_0 & 0 \end{pmatrix}, \qquad J := H_1 = \mathrm{diag}(1, 0, \ldots, 0, -1), \tag{176}$$

where

$$\vec{q} = (q_2, \ldots, q_{n+1})^T, \qquad \vec{p} = (p_2, \ldots, p_{n+1})^T.$$

Because of the specific choice of J these MNLS equations can be viewed as connected with the **BD.I** symmetric space. In most of this section we will use the simplest case with $n = 3$, though all results can be easily generalized for any rank of the algebra. The MNLS system itself takes the form

$$\mathrm{i}q_{2,t} + q_{2,xx} + 2q_2^2 p_2 + q_3^2 p_4 + 2q_3 q_2 p_3 = 0, \tag{177}$$

$$\mathrm{i}q_{3,t} + q_{3,xx} + 2q_2 q_4 p_3 + 2q_2 p_2 q_3 + 2q_3 q_4 p_4 + q_3^2 p_3 = 0, \tag{178}$$

$$\mathrm{i}q_{4,t} + q_{4,xx} + 2q_4^2 p_4 + q_3^2 p_2 + 2q_3 q_4 p_3 = 0, \tag{179}$$

$$\mathrm{i}p_{2,t} - p_{2,xx} - 2p_2^2 q_2 - p_3^2 q_4 - 2p_3 p_2 q_3 = 0, \tag{180}$$

$$\mathrm{i}p_{3,t} - p_{3,xx} - 2p_2 p_4 q_3 - 2p_2 q_2 p_3 - 2p_4 q_4 p_3 - p_3^2 q_3 = 0, \tag{181}$$

$$\mathrm{i}p_{4,t} - p_{4,xx} - 2p_4^2 q_4 - p_3^2 q_2 - 2p_3 p_4 q_3 = 0. \tag{182}$$

Its one-soliton solution derived via dressing procedure reads

$$q_k = \frac{\lambda^- - \lambda^+}{\Delta}\left(\mathrm{e}^{-\mathrm{i}\lambda^+(x+\lambda^+ t)} n_{0,1} m_{0,k} + (-1)^k\,\mathrm{e}^{-\mathrm{i}\lambda^-(x+\lambda^- t)} n_{0,6-k} m_{0,5}\right), \tag{183}$$

$$p_k = \frac{\lambda^- - \lambda^+}{\Delta}\left(\mathrm{e}^{\mathrm{i}\lambda^-(x+\lambda^- t)} n_{0,k} m_{0,1} + (-1)^k\,\mathrm{e}^{\mathrm{i}\lambda^+(x+\lambda^+ t)} n_{0,5} m_{0,6-k}\right), \tag{184}$$

where

$$\Delta = \mathrm{e}^{\mathrm{i}(\lambda^- - \lambda^+)(x+(\lambda^- + \lambda^+)t)} n_{0,1} m_{0,1} + \sum_{k=2}^{4} n_{0,k} m_{0,k} + \mathrm{e}^{\mathrm{i}(\lambda^+ - \lambda^-)(x+(\lambda^- + \lambda^+)t)} n_{0,5} m_{0,5}.$$

Consider a $\mathbb{Z}_2$ reduction of the form

$$K U^\dagger(x, \lambda^*) K^{-1} = U(x, \lambda), \qquad \text{i.e.} \qquad K q^\dagger K^{-1} = q, \qquad K J^* K^{-1} = J. \tag{185}$$

If J is real there are two inequivalent choices for K satisfying $KJK^{-1} = J$: the first one is the typical reduction via an element of the Cartan subgroup, $K_1 = \mathrm{diag}(\epsilon_1, \epsilon_2, 1, \epsilon_2, \epsilon_1)$, $\epsilon_{1,2}^2 = 1$; the second one is a reduction via a Weyl reflection S_{e_2}, see (171). The reductions obtained with K_1 and K_2 respectively give

$$p_2 = \epsilon_1 \epsilon_2 q_2^*, \qquad p_3 = \epsilon_1 q_3^*, \qquad p_4 = \epsilon_1 \epsilon_2 q_4^*; \tag{186}$$

$$p_2 = q_4^*, \qquad p_3 = -q_3^*, \qquad p_4 = q_2^*. \tag{187}$$

The former $\mathbb{Z}_2$ reduction leads to the following three-component system of NLS equation:

$$\mathrm{i}q_{2,t} + q_{2,xx} + 2\epsilon_1(\epsilon_2|q_2|^2 + |q_3|^2)q_2 + \epsilon_1\epsilon_2 q_3^2 q_4^* = 0, \tag{188}$$

$$\mathrm{i}q_{3,t} + q_{3,xx} + 2\epsilon_1 q_2 q_4 q_3^* + \epsilon_1(2\epsilon_2|q_2|^2 + 2\epsilon_2|q_4|^2 + |q_3|^2)q_3 = 0, \tag{189}$$

$$\mathrm{i}q_{4,t} + q_{4,xx} + 2\epsilon_1(\epsilon_2|q_4|^2 + |q_3|^2)q_4 + \epsilon_1\epsilon_2 q_3^2 q_2^* = 0. \tag{190}$$

In order to integrate this system we apply the dressing procedure with the dressing factor (38). Taking into account that in the reduced case we have the relations

$$\lambda^+ = (\lambda^-)^* = \mu + \mathrm{i}\nu, \qquad |m_0\rangle = K_1|n_0^*\rangle, \tag{191}$$

we find that its soliton solution is given by

$$q_2 = \frac{-2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}(\epsilon_2\,\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,2}^* + \epsilon_1\,\mathrm{e}^{-\nu(x-ut)}n_{0,4}n_{0,5}^*), \tag{192}$$

$$q_3 = \frac{-2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}(\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,3}^* - \epsilon_1\,\mathrm{e}^{-\nu(x-ut)}n_{0,3}n_{0,5}^*), \tag{193}$$

$$q_4 = \frac{-2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}(\epsilon_2\,\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,4}^* + \epsilon_1\,\mathrm{e}^{-\nu(x-ut)}n_{0,2}n_{0,5}^*), \tag{194}$$

$$\Delta = \epsilon_1\,\mathrm{e}^{2\nu(x-ut)}|n_{0,1}|^2 + \epsilon_2(|n_{0,2}|^2 + |n_{0,4}|^2) + |n_{0,3}|^2 + \epsilon_1\,\mathrm{e}^{-2\nu(x-ut)}|n_{0,5}|^2, \tag{195}$$

$$v = \frac{\nu^2 - \mu^2}{\mu}, \qquad u = -2\mu. \tag{196}$$

The latter reduction gives rise to another inequivalent system of three NLS equations

$$\mathrm{i}q_{2,t} + q_{2,xx} + 2(q_2 q_4^* - |q_3|^2)q_2 + q_3^2 q_2^* = 0, \tag{197}$$

$$\mathrm{i}q_{3,t} + q_{3,xx} - 2q_2 q_4 q_3^* + (2q_2 q_4^* + 2q_4 q_2^* - |q_3|^2)q_3 = 0, \tag{198}$$

$$\mathrm{i}q_{4,t} + q_{4,xx} + 2(q_4 q_2^* - |q_3|^2)q_4 + q_3^2 q_4^* = 0. \tag{199}$$

Then we have the following one-soliton solution:

$$q_2 = \frac{-2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}(\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,4}^* + \mathrm{e}^{-\nu(x-ut)}n_{0,4}n_{0,5}^*), \tag{200}$$

$$q_3 = \frac{2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}(\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,3}^* + \mathrm{e}^{-\nu(x-ut)}n_{0,3}n_{0,5}^*), \tag{201}$$

$$q_4 = \frac{-2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}(\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,2}^* + \mathrm{e}^{-\nu(x-ut)}n_{0,2}n_{0,5}^*), \tag{202}$$

$$\Delta = \mathrm{e}^{2\nu(x-ut)}|n_{0,1}|^2 + (n_{0,2}n_{0,4}^* + n_{0,2}^*n_{0,4}) - |n_{0,3}|^2 + \mathrm{e}^{-2\nu(x-ut)}|n_{0,5}|^2. \tag{203}$$

As in the previous example the polarization vectors are interrelated via (191) which in this case reads

$$m_{0,k} = \begin{cases} n_{0,k}^*, & k = 1, 5 \\ (-1)^k n_{0,6-k}^*, & k = 2, 3, 4. \end{cases}$$

Next we consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction, which is a combination of reductions with K_1 and K_2. This is possible only for $\epsilon_1 = -1$. Then

$$p_{2,4} = -\epsilon_2 q_{2,4}^*, \qquad q_2 = -\epsilon_2 q_4, \qquad p_3 = -q_3^*, \tag{204}$$

and we obtain the following two-component NLS system:

$$\mathrm{i}q_{2,t} + q_{2,xx} - 2(\epsilon_2|q_2|^2 + |q_3|^2)q_2 + q_3^2 q_2^* = 0, \tag{205}$$

$$\mathrm{i}q_{3,t} + q_{3,xx} - (4\epsilon_2|q_2|^* + |q_3|^2)q_3 + 2\epsilon_2(q_2)^2 q_3^* = 0. \tag{206}$$

Its one-soliton solution takes the form

$$q_2 = \frac{2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}\epsilon_2(\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,2}^* + \mathrm{e}^{-\nu(x-ut)}n_{0,2}n_{0,5}^*), \tag{207}$$

$$q_3 = \frac{2\mathrm{i}\nu}{\Delta}\,\mathrm{e}^{-\mathrm{i}\mu(x-vt)}(\mathrm{e}^{\nu(x-ut)}n_{0,1}n_{0,3}^* + \mathrm{e}^{-\nu(x-ut)}n_{0,3}n_{0,5}^*), \tag{208}$$

$$\Delta = \mathrm{e}^{2\nu(x-ut)}|n_{0,1}|^2 - 2\epsilon_2|n_{0,2}|^2 - |n_{0,3}|^2 + \mathrm{e}^{-2\nu(x-ut)}|n_{0,5}|^2, \tag{209}$$

$$v = \frac{\nu^2-\mu^2}{\mu}, \qquad u = -2\mu, \tag{210}$$

where we have made use of the following relations:

$$m_{0,k} = \begin{cases} n_{0,k}^*, & k = 1, 5 \\ -\epsilon_2 n_{0,k}^* = n_{0,6-k}^*, & k = 2, 4, \\ -n_{0,3}^*, & k = 3. \end{cases}$$

5.4. *MMKdV equations on symmetric spaces of* **BD.I** *type*

Multicomponent MKdV related to $so(n+2)$ algebra possesses a Lax pair with the L operator (173) and M-operator in the form

$$\begin{aligned} &M\psi(x,t,\lambda) \equiv \mathrm{i}\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - \lambda^3 J)\psi(x,t,\lambda), \\ &V_2(x,t) = q(x,t), \qquad V_1(x,t) = \mathrm{i}\,\mathrm{ad}_J^{-1}\partial_x q + \tfrac{1}{2}\left[\mathrm{ad}_J^{-1}q, q(x,t)\right], \\ &V_0(x,t) = -\partial_{xx}^2 q + \tfrac{1}{2}\left[\mathrm{ad}_J^{-1}q, \left[\mathrm{ad}_J^{-1}q, q(x,t)\right]\right] + \mathrm{i}[\partial_x q, q]. \end{aligned} \tag{211}$$

The corresponding MMKdV equations can be written down in compact form as

$$\partial_t\vec{q} + \partial_{xxx}^3\vec{q} + 3(\vec{p},\vec{q})\partial_x\vec{q} + 3(\partial_x\vec{q},\vec{p})\vec{q} - 3(\partial_x\vec{q}s_0\vec{q})s_0\vec{p} = 0, \tag{212}$$

$$\partial_t\vec{p} + \partial_{xxx}^3\vec{p} + 3(\vec{p},\vec{q})\partial_x\vec{p} + 3(\partial_x\vec{p},\vec{q})\vec{p} - 3\,(\partial_x\vec{p}s_0\vec{p})\,s_0\vec{q} = 0. \tag{213}$$

Consider a $\mathbb{Z}_2$ reduction of the type

$$U^\dagger(\lambda^*) = U(\lambda), \qquad \Rightarrow \qquad \vec{p} = \vec{q}^*. \tag{214}$$

Then we obtain the reduced system of equations

$$\partial_t\vec{q} + \partial_{xxx}^3\vec{q} + 3|\vec{q}|^2\partial_x\vec{q} + 3(\partial_x\vec{q},\vec{q}^*)\vec{q} - 3\,(\partial_x\vec{q}s_0\vec{q})\,s_0\vec{q}^* = 0. \tag{215}$$

Its one-soliton solution reads

$$\begin{aligned} &q_k = \frac{-\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\mu(x-ut-\delta_0)}}{\cosh(2\nu(x-vt-\xi_0)) + \mathcal{C}}(\mathrm{e}^{\nu(x-vt-\xi_0)}c_k^* + (-1)^k\,\mathrm{e}^{-\nu(x-vt-\xi_0)}c_{n+3-k}), \\ &c_k = \frac{n_{0,k}}{\sqrt{|n_{0,1}||n_{0,n+2}|}}, \qquad k = 2,\ldots,n+1 \qquad \mathcal{C} = \sum_{k=2}^{n+1}\frac{|n_{0,k}|^2}{2|n_{0,1}||n_{0,n+2}|}, \\ &v = \nu^2 - 3\mu^2, \qquad u = 3\nu^2 - \mu^2, \qquad \delta_0 = \frac{\arg n_{0,1}}{\mu}, \qquad \xi_0 = \frac{1}{2\nu}\ln\frac{|n_{0,n+2}|}{|n_{0,1}|}, \end{aligned} \tag{216}$$

provided we have fixed $\arg n_{0,1} = -\arg n_{0,n+2}$ by using the natural $U(1)$ symmetry of the solution.

As a final example let us consider MKdV related to $so(5)$ with a $\mathbb{Z}_2$ reduction of the type

$$KU^\dagger(-\lambda^*)K^{-1} = U(\lambda) \qquad \Rightarrow \qquad Kq^\dagger K^{-1} = q, \qquad KJK^{-1} = -J. \tag{217}$$

We choose $K = K_0 \circ W_{e_1}$ where

$$W_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Rightarrow \qquad K = \begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon_1 \\ 0 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 & 0 \\ -\epsilon_1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{218}$$

is the Weyl reflection with respect to the hyperplane orthogonal to e_1. The following interrelations hold true:

$$q_4 = -\epsilon_1\epsilon_2 q_2^*, \qquad q_3 = -\epsilon_1 q_3^*, \qquad p_4 = -\epsilon_1\epsilon_2 p_2^*, \qquad p_3 = -\epsilon_1 p_3^*. \tag{219}$$

As a consequence of the reduction we have

$$\lambda^+ = -(\lambda^-)^*, \qquad |m\rangle = K|n\rangle^*, \tag{220}$$

or

$$(\lambda^\pm)^* = -\lambda^\pm, \qquad |n\rangle = SK|n\rangle^*, \qquad \langle m| = \langle m|^*(SK)^{-1}. \tag{221}$$

Applying another $\mathbb{Z}_2$ reduction of the type

$$U^T(-\lambda) = -U(\lambda), \qquad \Rightarrow \qquad q^T = -q, \tag{222}$$

we obtain that

$$\lambda^+ = -\lambda^-, \qquad |m\rangle = |n\rangle. \tag{223}$$

The corresponding system of MKdV is

$$q_{2,t} + q_{2,xxx} - 3(q_2q_3)_x q_3 + 3\epsilon_1\epsilon_2 q_3 q_2^* q_{3,x} - 6q_2^2 q_{2,x} = 0, \tag{224}$$

$$q_{3,t} + q_{3,xxx} + 3\epsilon_1\epsilon_2 |q_2|_x^2 q_3 - 3(q_2q_3)_x q_2 - 3(q_2^* q_3)_x q_2^* - 3q_3^2 q_{3,x} = 0, \tag{225}$$

and is new to the best of our knowledge. Combining both reductions we again may have two types of solitons. The doublet soliton corresponds to purely imaginary $\lambda^\pm = \pm \mathrm{i}\nu$ and $|n\rangle = SK|n\rangle^*$ and is given by

$$q_2 = \frac{\mathrm{i}\nu\, \mathrm{e}^{\mathrm{i}\delta_0}}{\epsilon_1 \cosh 2\nu(x - vt - \xi_0) + \mathcal{C}} \left(\mathrm{e}^{\nu(x-vt-\xi_0)} c_2 + \mathrm{e}^{-\nu(x-vt-\xi_0)} c_4\right), \tag{226}$$

$$q_3 = \frac{2\mathrm{i}\nu c_3\, \mathrm{e}^{\mathrm{i}\delta_0} \sinh \nu(x - vt - \xi_0)}{\epsilon_1 \cosh 2\nu(x - vt - \xi_0) + \mathcal{C}},$$

$$\delta_0 = \arg n_{0,1} = \arg n_{0,5} = \frac{l\pi}{2}, \qquad l \in \mathbb{Z}, \qquad \mathcal{C} = \frac{2\epsilon_2 \operatorname{Re}(n_{0,2}n_{0,4}) + |n_{0,3}|^2}{2|n_{0,1}||n_{0,5}|}, \tag{227}$$

$$c_1^* = -\epsilon_1 c_1, \qquad c_2^* = -\epsilon_2 c_4, \qquad c_3^* = -c_3, \qquad c_5^* = -\epsilon_1 c_5,$$

where c_k, v and ξ_0 coincide with those in the previous example when $r = 2$. From (219) it follows that q_3 is either real or purely imaginary valued function.

Yet another possibility to ensure compatibility of both $\mathbb{Z}_2$ reductions is to modify the dressing factor into

$$u(x, t, \lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - \lambda_0} - \frac{KSA^*(x,t)SK}{\lambda + \lambda_0^*} - \frac{SA(x,t)S}{\lambda + \lambda_0} + \frac{KA^*(x,t)K}{\lambda - \lambda_0^*}, \tag{228}$$

which leads to the quadruplet soliton solution

$$q(x, t) = [J, A - KSA^*SK - SAS + KA^*K](x, t). \tag{229}$$

In order to find it we have to calculate the matrix $A(x,t)$. For that purpose it proves to be convenient to decompose A into $A = XF^T$, where X and F are rectangular matrices of rank $s \leqslant r$ and $\lambda_0 = \mu + \mathrm{i}\nu$. It can be checked that $F(x,t) = \mathrm{e}^{\mathrm{i}(\lambda_0 x + \lambda_0^3 t)J} F_0$, $F_0 = \text{const}$. In the simplest $s = 1$ case for the factor X one can obtain the following:

$$X = \frac{a^* F + bSKF^* - cKF^*}{|a|^2 + b^2 - c^2},$$

where

$$a = \frac{F^T F}{2\lambda_0} = \frac{|F_{0,1}F_{0,5}|}{\lambda_0}(\cosh 2(\phi_{\mathrm{R}} - \mathrm{i}\phi_{\mathrm{I}}) + \mathcal{C}_a), \qquad \mathcal{C}_a = \frac{F_{0,2}^2 + F_{0,3}^2 + F_{0,4}^2}{2|F_{0,1}F_{0,5}|},$$

$$b = \frac{(F^\dagger SKF)}{2\mathrm{i}\nu} = \frac{\mathrm{i}|F_{0,1}F_{0,5}|}{\nu}(\cosh 2\phi_{\mathrm{R}} + \mathcal{C}_b), \qquad \mathcal{C}_b = \frac{2\mathrm{Re}(F_{0,2}^* F_{0,4}) + |F_{0,3}|^2}{2|F_{0,1}F_{0,5}|},$$

$$c = \frac{(F^\dagger KF)}{2\mu} = \frac{|F_{0,1}F_{0,5}|}{\mu}(\cos 2\phi_{\mathrm{I}} + \mathcal{C}_c), \qquad \mathcal{C}_c = \frac{|F_{0,2}|^2 - |F_{0,3}|^2 + |F_{0,4}|^2}{2|F_{0,1}F_{0,5}|},$$

$$\phi_{\mathrm{R}} = \nu\left(x - vt - \frac{1}{2\nu}\ln\frac{|F_{0,1}|}{|F_{0,5}|}\right), \qquad \phi_{\mathrm{I}} = \mu\left(x - ut - \frac{\arg F_{0,5}}{\mu}\right),$$

$$\arg F_{0,1} = -\arg F_{0,5}.$$

Substituting this result into (229) and choosing $\epsilon_1 = \epsilon_2 = 1$ one derives

$$\begin{aligned} q_2 = \frac{2\sqrt{|F_{0,1}F_{0,2}F_{0,4}F_{0,5}|}}{|a|^2 + b^2 - c^2} \big\{ & a^* \cosh(\phi_{\mathrm{R}}^- - \mathrm{i}\phi_{\mathrm{I}}^-) - b\big(\cosh\big(\phi_{\mathrm{R}}^- + \mathrm{i}\phi_{\mathrm{I}}^+\big) + \cosh\big(\phi_{\mathrm{R}}^+ - \mathrm{i}\phi_{\mathrm{I}}^-\big)\big) \\ & - a\cosh\big(\phi_{\mathrm{R}}^+ + \mathrm{i}\phi_{\mathrm{I}}^+\big) + c\big(\cosh\big(\phi_{\mathrm{R}}^+ + \mathrm{i}\phi_{\mathrm{I}}^-\big) - \cosh\big(\phi_{\mathrm{R}}^- - \mathrm{i}\phi_{\mathrm{I}}^+\big)\big)\big\}, \end{aligned} \qquad (230)$$

$$q_3 = \frac{2\mathrm{i}\sqrt{|F_{0,1}F_{0,5}|}}{|a|^2 + b^2 - c^2}\,\mathrm{Im}\{(b + c)\sinh(\phi_{\mathrm{R}} + \mathrm{i}\phi_{\mathrm{I}}) - a^*\sinh(\phi_{\mathrm{R}} - \mathrm{i}\phi_{\mathrm{I}})\}F_{0,3}, \qquad (231)$$

where we have used the following notation:

$$\phi_{\mathrm{R}}^\pm = \phi_{\mathrm{R}} \pm \frac{1}{2}\ln\frac{|F_{0,2}|}{|F_{0,4}|}, \qquad \phi_{\mathrm{I}}^\pm = \phi_{\mathrm{I}} \pm \arg F_{0,4}. \qquad (232)$$

6. Discussion and further studies

Here we shall outline some further topics which could be studied and which could lead to a deeper understanding of these soliton properties.

The first obvious remark is that $sl(n)$ contains as subalgebras also $so(p)$ and $sp(p)$ subalgebras. So it will be interesting to specify the conditions under which $L(\lambda)$ has solitons of type $so(p)$ or $sp(p)$.

The second remark of the same nature is that one can start with $L(\lambda)$ related to any simple Lie algebra $\mathfrak{g}$ (e.g., $so(n)$ or $sp(n)$). The analysis of soliton solutions for NLEE related to such systems require deeper knowledge of Lie algebras and their representations.

The explicit form of the corresponding N-wave system related to these algebras has been reported in [4, 5, 13], see also [29, 30]. What could be done is to analyze the structure of its soliton solutions [29, 30] which are more involved, especially in the case when additional reductions are imposed. The construction of the corresponding dressing factors is more complicated, but one can expect that new interesting types of integrable cubic and quartic interactions could be obtained.

Another important question is the study of solitons constructed by projectors of rank 2 and higher. Such solitons have been already used by Wadati *et al* [31] in describing BEC with hyperfine structure $F = 1$. Now each soliton will be parameterized by two polarization vectors; the corresponding eigensubspace will be two dimensional too. Among the various types of rank-2 one-soliton solutions, there will be various possible configurations for the two polarization vectors. An example of a dressing factor $u(x, t, \lambda)$ constructed by a projector of second rank is the following one:

$$u(x,t,\lambda) = \mathbb{1} + \left(\frac{1}{\sqrt{c(\lambda)}} - 1\right) P_2 + (\sqrt{c(\lambda)} - 1)\overline{P}_2.$$

Such type of a dressing factor was used in [32] to derive the soliton solutions to a multicomponent Schrödinger equation relate to symplectic algebra $sp(4)$.

It is known in general how the machinery, well understood for the AKNS system such as Wronskian relations, expansions over 'squared solutions', etc can be generalized also for these types of systems. The dressing method, after some modifications, can also be applied, leading to the derivation of their soliton solutions.

An interesting problem is the study of how the different possible reductions (see, e.g., [29]) of these systems will influence the number of one-soliton types.

Soliton interactions for various different types of solitons of these systems also present interesting problems. From the results known for the N-wave systems [9, 10] it is known that new effects in soliton interaction, such as soliton decay and soliton fusion may arise.

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Appendix A. The orthogonal algebras and BD.I symmetric spaces

The definition of orthogonality used in equations (159)–(161) has the advantage that the Cartan subalgebras can be represented by diagonal matrices.

It is well known that the two cases of odd and even n are substantially different. The algebras $so(2r+1)$ are known as the $\mathbf{B}_r$ series according to the Cartan classification [33]. Assuming that the reader is familiar with the basics of simple Lie algebras here we just recall that the root systems of these algebras consist of $\Delta^- \cup \Delta^+$ where $\Delta^- = -\Delta^+$ and $\Delta^+ \equiv \{e_i \pm e_j, e_j\}$ with $1 \leqslant i < j \leqslant r$. We provide also the Cartan–Weyl basis of these algebras in the typical $(2r+1)$-dimensional representation

$$\begin{aligned} E_{e_i-e_j} &\equiv E_{ij} - S_0 E_{ji} S_0 = E_{i,j} - (-1)^{i+j} E_{\bar{j},\bar{i}}, \\ E_{e_i+e_j} &\equiv E_{i\bar{j}} - S_0 E_{\bar{j}i} S_0 = E_{i,\bar{j}} - (-1)^{i+j} E_{j,\bar{i}}, \\ E_{e_i} &\equiv E_{i,r+1} - S_0 E_{r+1,i} S_0 = E_{i,r+1} - (-1)^{i+r+1} E_{r+1,\bar{i}}, \\ H_{e_j} &\equiv E_{jj} - S_0 E_{j,j} S_0 = E_{j,j} - E_{\bar{j},\bar{j}}. \qquad E_{-\alpha} = E_\alpha^T. \end{aligned} \tag{A.1}$$

Each symmetric space is obtained by applying a Cartan involution. This involution splits the group $SO(2r+1)$ into a subgroup $SO(2r-1) \otimes SO(2)$ and a factor space $SO(2r+1)/SO(2r-1) \otimes SO(2)$. Effectively the system of positive roots is split into $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where the subsets of roots are defined as follows:

$$\Delta_0^+ = \{e_i \pm e_j, \quad e_i, \quad 2 \leqslant i < j \leqslant r\}, \qquad \Delta_1^+ = \{e_1 \pm e_j, \quad e_1, \quad 2 \leqslant j \leqslant r\}. \tag{A.2}$$

In fact Δ_0^+ contains all positive roots that are orthogonal to e_1, while Δ_1^+ contains all positive roots that have scalar product equal to 1 with e_1.

Similarly one can consider the algebras $so(2r)$ which are known as the $\mathbf{D}_r$ series. Their Cartan–Weyl basis in the typical $2r$-dimensional representation is given by

$$\begin{aligned} E_{e_i-e_j} &\equiv E_{ij} - S_0 E_{ji} S_0 = E_{i,j} - (-1)^{i+j} E_{\bar{j},\bar{i}}, \\ E_{e_i+e_j} &\equiv E_{i\bar{j}} - S_0 E_{\bar{j}i} S_0 = E_{i,\bar{j}} - (-1)^{i+j} E_{j,\bar{i}}, \\ H_{e_j} &\equiv E_{jj} - S_0 E_{j,j} S_0 = E_{j,j} - E_{\bar{j},\bar{j}}. \qquad E_{-\alpha} = E_\alpha^T. \end{aligned} \tag{A.3}$$

The corresponding symmetric space is $SO(2r+2)/SO(2r)\otimes SO(2)$. The system of positive roots is split into $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where

$$\Delta_0^+ = \{e_i \pm e_j, \quad 2 \leqslant i < j \leqslant r\}, \qquad \Delta_1^+ = \{e_1 \pm e_j, \quad 2 \leqslant j \leqslant r\}. \tag{A.4}$$

Again Δ_0^+ contains all positive roots that are orthogonal to e_1, while Δ_1^+ contains all positive roots that have scalar product equal to 1 with e_1.

Appendix B. Higher spin representations of $sl(2)$

Here we construct the $sl(2)$ dressing factor (130) for higher spin representation of $sl(2)$. This we do by using completely symmetric tensor powers of u.

Using the way how u acts on basic vectors e_i, $i = 1, 2$ in $\mathbb{C}_2$

$$ue_1 = u_{11}e_1 + u_{21}e_2, \qquad ue_2 = u_{12}e_1 + u_{22}e_2. \tag{B.1}$$

The normalized basis in the completely symmetrized tensor product of $\mathbb{C}^2 \odot \mathbb{C}^2$ is given by

$$\epsilon_1 = e_1 \otimes e_1, \qquad \epsilon_2 = \frac{1}{\sqrt{2}}(e_2 \otimes e_1 + e_1 \otimes e_2), \qquad \epsilon_3 = e_2 \otimes e_2. \tag{B.2}$$

Since $u(x, t, \lambda)$ belongs to the group $SL(2)$ it must act on the basic elements as follows:

$$U^{(3)}(x, \lambda)(e_i \otimes e_j) = (ue_i) \otimes (ue_j). \tag{B.3}$$

Thus we obtain the following representation for the dressing factor for spin-1 representation

$$U^{(3)} \equiv u \odot u = \begin{pmatrix} u_{11}^2 & \sqrt{2}u_{11}u_{12} & u_{12}^2 \\ \sqrt{2}u_{11}u_{21} & u_{11}u_{22} + u_{12}u_{21} & \sqrt{2}u_{22}u_{12} \\ u_{21}^2 & \sqrt{2}u_{22}u_{21} & u_{22}^2 \end{pmatrix}. \tag{B.4}$$

Now we have to insert the expressions for u_{ij} in terms of c_1, n_k and m_k and thus we derive

$$U^{(3)} \equiv u \odot u = c_1 \pi_1^{(3)} + \pi_0^{(3)} + \frac{1}{c_1}\pi_{-1}^{(3)}, \tag{B.5}$$

where the projectors π_a, $a = -1, 0, 1$ are all rank-1 projectors of the form

$$\pi_1^{(3)} = \frac{|N_1^{(3)}\rangle\langle M_1^{(3)}|}{\langle M_1^{(3)}|N_1^{(3)}\rangle}, \qquad \pi_0^{(3)} = \frac{|N_0^{(3)}\rangle\langle M_0^{(3)}|}{\langle M_0^{(3)}|N_0^{(3)}\rangle}, \qquad \pi_{-1}^{(3)} = \frac{|N_{-1}^{(3)}\rangle\langle N_{-1}^{(3)}|}{\langle M_{-1}^{(3)}|N_{-1}^{(3)}\rangle}, \tag{B.6}$$

where $\langle M_1^{(3)}|N_1^{(3)}\rangle = \langle M_0^{(3)}|N_0^{(3)}\rangle = \langle M_{-1}^{(3)}|N_{-1}^{(3)}\rangle = (m_1n_1 + m_2n_2)^2$

$$\begin{gathered} |N_1^{(3)}\rangle = \begin{pmatrix} n_1^2 \\ \sqrt{2}n_1n_2 \\ n_2^2 \end{pmatrix}, \qquad |N_{-1}^{(3)}\rangle = \begin{pmatrix} m_2^2 \\ -\sqrt{2}m_1m_2 \\ m_1^2 \end{pmatrix}, \qquad |N_0^{(3)}\rangle = \begin{pmatrix} \sqrt{2}m_2n_1 \\ n_2m_2 - n_1m_1 \\ -\sqrt{2}n_2m_1 \end{pmatrix}, \\ \langle M_1^{(3)}| = (m_1^2, \sqrt{2}m_1m_2, m_2^2), \qquad \langle M_{-1}^{(3)}| = (n_2^2, -\sqrt{2}n_1n_2, n_1^2), \\ \langle M_0^{(3)}| = (\sqrt{2}n_2m_1, n_2m_2 - n_1m_1, -\sqrt{2}n_1m_2). \end{gathered} \tag{B.7}$$

Similarly the normalized basis in the completely symmetrized tensor products of $\mathbb{C}^2 \odot \mathbb{C}^2 \odot \mathbb{C}^2$ is given by

$$\begin{aligned} \epsilon_1' &= e_1 \otimes e_1 \otimes e_1, \qquad \epsilon_2' = \frac{1}{\sqrt{3}}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1), \\ \epsilon_3' &= \frac{1}{\sqrt{3}}(e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1), \qquad \epsilon_4' = e_2 \otimes e_2 \otimes e_2. \end{aligned} \tag{B.8}$$

Therefore the dressing factor obtains the form

$$U^{(4)} = \begin{pmatrix} u_{11}^3 & \sqrt{3}u_{11}^2 u_{12} & \sqrt{3}u_{11}u_{12}^2 & u_{12}^3 \\ \sqrt{3}u_{11}^2 u_{21} & u_{11}^2 u_{22} + 2u_{11}u_{12}u_{21} & u_{12}^2 u_{21} + 2u_{11}u_{12}u_{22} & \sqrt{3}u_{12}^2 u_{22} \\ \sqrt{3}u_{11}u_{21}^2 & u_{12}u_{21}^2 + 2u_{11}u_{21}u_{22} & u_{11}u_{22}^2 + 2u_{12}u_{21}u_{22} & \sqrt{3}u_{12}u_{22}^2 \\ u_{21}^3 & \sqrt{3}u_{21}^2 u_{22} & \sqrt{3}u_{21}u_{22}^2 & u_{22}^3 \end{pmatrix}. \tag{B.9}$$

It can be decomposed into

$$U^{(4)} = \pi_{-3/2}^{(4)} c^{-3/2} + \pi_{-1/2}^{(4)} c^{-1/2} + \pi_{1/2}^{(4)} c^{1/2} + \pi_{3/2}^{(4)} c^{3/2} \tag{B.10}$$

where

$$\pi_a^{(4)} = \frac{\left|N_a^{(4)}\right\rangle\!\left\langle M_a^{(4)}\right|}{\left\langle M_a^{(4)}\middle|N_a^{(4)}\right\rangle}, \qquad a = -3/2, -1/2, 1/2, 3/2. \tag{B.11}$$

The (co) vectors $\left|N_a^{(4)}\right\rangle$ $(\langle m_a|)$ are given by

$$\begin{aligned} \left|N_{-3/2}^{(4)}\right\rangle &= \left(m_2^3, -\sqrt{3}m_2^2 m_1, \sqrt{3}m_2 m_1^2, -m_1^3\right)^T, \\ \left\langle M_{-3/2}^{(4)}\right. &= \left(n_2^3, -\sqrt{3}n_2^2 n_1, \sqrt{3}n_2 n_1^2, -n_1^3\right) \\ \left|N_{-1/2}^{(4)}\right\rangle &= \left(\sqrt{3}n_1 m_2^2, m_2(n_2 m_2 - 2n_1 m_1), m_1(n_1 m_1 - 2n_2 m_2), \sqrt{3}n_2 m_1^2\right)^T, \\ \left\langle M_{-1/2}^{(4)}\right| &= \left(\sqrt{3}n_2^2 m_1, n_2(n_2 m_2 - 2n_1 m_1), n_1(n_1 m_1 - 2n_2 m_2), \sqrt{3}n_1^2 m_2\right) \\ \left|N_{1/2}^{(4)}\right\rangle &= \left(\sqrt{3}n_1^2 m_2, -n_1(n_1 m_1 - 2n_2 m_2), n_2(n_2 m_2 - 2n_1 m_1), -\sqrt{3}n_2^2 m_1\right)^T, \\ \left\langle M_{1/2}^{(4)}\right| &= \left(\sqrt{3}n_2 m_1^2, m_1(n_1 m_1 - 2n_2 m_2), m_2(n_2 m_2 - 2n_1 m_1), -\sqrt{3}n_1 m_2^2\right) \\ \left|N_{3/2}^{(4)}\right\rangle &= \left(n_1^3, \sqrt{3}n_1^2 n_2, \sqrt{3}n_1 n_2^2, n_2^3\right)^T, \\ \left\langle M_{3/2}^{(4)}\right| &= \left(m_1^3, \sqrt{3}m_1^2 m_2, \sqrt{3}m_1 m_2^2, m_2^3\right). \end{aligned}$$

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